XVIII. *On certain Dimensional Properties of Matter in the Gaseous State.*

Part I. *Experimental Researches on Thermal Transpiration of Gases through Porous Plates and on the Laws of Transpiration and Impulsion, including an Experimental Proof that Gas is not a continuous Plenum.*

Part II. *On an Extension of the Dynamical Theory of Gas, which includes the Stresses, Tangential and Normal, caused by a varying Condition of Gas, and affords an Explanation of the Phenomena of Transpiration and Impulsion.*

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PART I.—(EXPERIMENTAL).

SECTION I.—INTRODUCTION.

1. THE motion of gases through minute channels, such as capillary tubes, porous plugs, and apertures in thin plates, has been the subject of much attention during the last fifty years. The experimental study of these motions, principally by GRAHAM,* resulted in the discovery of several important properties of gases. And it is largely, if not mainly, as affording an explanation of these properties that the molecular theory has obtained such general credence.

It does not appear, however, that either the experimental investigations of these motions or the theoretical explanations of the properties revealed have hitherto been in any sense complete.

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* Edin. Phil. Trans., 1831; Phil. Trans., 1846 and 1863.

There exists a whole class of very marked phenomena which have escaped the notice of GRAHAM and subsequent observers; while several of the most marked and important facts discovered by GRAHAM have hitherto remained unconnected by any theory.

2. Amongst the best known of the phenomena is the difference in the rates at which different gases transpire through minute channels, and the consequent difference of pressure which ensues when two different gases initially at the same pressure are separated by a porous plate. It does not appear, however, that hitherto any attempt has been made to ascertain the existence of what may be considered a closely analogous phenomenon—that a difference of temperature on the two sides of the plate might cause gas, without any initial difference of pressure or any difference in chemical constitution, to pass through the plate—nor am I aware that such a result from a difference of temperature has been in any way surmised (see Appendix, note 1).

I have, however, now ascertained, by experiments which will be described at length, that a difference of temperature may be a very potent cause of transpiration through porous plates. So much so that with hydrogen on both sides of a porous plate, the pressure on one side being that of the atmosphere, a difference of 160° F. (from 52° to 212°) in the temperature on the two sides of the plate secured a permanent difference in the pressure on the two sides equal to an inch of mercury; the higher pressure being on the hotter side. With different gases and different plates various results were obtained, which are however, as will be seen, connected by definite laws.

I propose to call the motion of the gas caused by a difference of temperature *Thermal Transpiration* (see Appendix, note 2).

3. Again, although GRAHAM found that he obtained not only very different results but also very different laws of motion with plates of different coarseness or with plates and capillary tubes,[*] neither he nor any subsequent observer appears to have followed up this lead. As regards GRAHAM this appears to me to be somewhat surprising. For although he may have considered the mere difference in the results to have been analogous to the difference found by POISEIULLE for liquids, it would seem as though the difference in the laws of motion which he obtained should have excited his curiosity; and then, as he was avowedly of opinion that gas is molecular, he could hardly have failed to observe that so long as the distance separating the molecules in the gas bore a fixed relation to the breadth of the openings in his plates he should have had the same laws of motion. This view, however, appears to have escaped him as well as all subsequent observers. Otherwise it would have been seen that with a simple gas such as hydrogen, similar results must be obtained so long as the density of the gas is inversely proportional to the lateral dimensions of the passages through the plates.

By experiments, to be described, I have now fully established this law. I find that with different plates similar results are obtained when the densities of the gas with

[*] Phil. Trans., 1863.

the different plates bear a fixed ratio; and this is the case whatever may be the cause of the transpiration, *i.e.*, a difference of temperature or a difference of pressure (a difference of gas I have not investigated, as it was obviously unnecessary to do so). Thus with two plates, one of stucco and the other of meerschaum, similar results of transpiration caused by pressure were obtained when the densities with the plates were respectively as 1 to 5·6, both with hydrogen and air and at pressures ranging from 30 to 2 inches of mercury. Also with the same two plates similar results of thermal transpiration were obtained when the densities were respectively as 1 to 6·5 both for air and hydrogen, and through a range of pressures from 30 to ·25 inches of mercury. The discrepancies of 5·6 and 6·5 were in all probability owing to a slightly altered condition of the plates (see Appendix, note 4).

This correspondence of the results at corresponding densities holds, although the law of motion changes. Thus with air at 30 inches the law was the same as that obtained by GRAHAM for stucco plates, while at the smallest pressures (·25 inch) it was nearly the same as he found for graphite plates or apertures in thin plates.

4. Having established this law of corresponding results at corresponding densities, it became apparent that the results obtained with plates of different coarseness and with the same plates but different densities of gas also followed a definite law. This law, which admits of symbolical expression, shows that there exists a definite relation between the results obtained, the lateral dimensions of the passages, and the density of the gas.

This law is important as reconciling results which have hitherto appeared to be discordant, such as GRAHAM'S results with plates of different coarseness, and as tending to complete the experimental investigation; but it has another and a more general importance.

It may not appear at first sight, but on consideration it will be seen that this law amounts to nothing less than an absolute experimental demonstration that gas possesses a heterogeneous structure—that it is not a continuous plenum of which each part into which it may be divided has the same properties as the whole.

It would appear that GRAHAM must have had this proof, so to speak, under his eyes, and it is strange that both he and subsequent observers have overlooked it. It seems possible, however, that they were not alive to the importance of such a demonstration. It is now so generally assumed that gas does possess molecular structure that the weakness of the evidence on which the assumption is based and the importance of further proof are points that are apt to escape notice.

The importance of an experimental demonstration that gas possesses molecular structure.

5. The idea of molecular gas does not appear to have originated from the recognition of properties of gas which were inconsistent with the idea of a continuous plenum, but from a wish to reconcile the properties of gas with the properties of other substances,

or more strictly with some general property of matter. And the general conviction which may be said to prevail at the present time is owing to the simplicity of the assumptions on which the molecular hypothesis is based, and the completeness with which many of the properties of gases have been shown to follow from the molecular hypothesis.

But it will be readily seen that however simple may be the assumptions of the kinetic theory, and however completely the properties of gases may be shown to follow from these assumptions, this is no disproof of the possibility that gas may be a continuous substance, each elementary portion of which is endowed with all the properties of the whole, and unless this is disproved there may exist doubt as to the necessity for the kinetic theory.

Any direct proof, therefore, that gas is not ultimately continuous altogether alters the position of the molecular hypothesis.

The sufficiency of the demonstration that gas is not structureless.

6. In order to prove that gas is not continuous it is not necessary that we should be able to perceive the actual structure; we have only to find some property of a certain quantity of gas which can be shown not to be possessed by all the parts—some property which is altered by a re-arrangement of the parts.

Hitherto I believe that no such property has been recognised, or at all events the conclusions to be drawn from such a property have not been recognised. The phenomena of transpiration as well as those of the radiometer depend on such properties, but these properties have not been sufficiently understood to bring out the conclusion. This conclusion however follows directly from the law indicated in Art. 4, viz.: that the results of transpiration and impulsion depend on the relation between the size of the internal objects and the density of the gas.

The force of this reasoning will be better seen after the results of the experiments have been described, but it is introduced here to show the importance which attaches to what otherwise might be considered secondary properties of gases.

To these properties I must now return, not having yet indicated how I was led to make the experiments, and besides those already mentioned there remains an important class of phenomena to be noticed.

The results deduced from theory.

7. Although the existence of the phenomena of thermal transpiration and the existence of the law of corresponding results at corresponding densities have been verified by experiment they were not so discovered.

They followed from what appeared to me to be a successful attempt to complete the explanation I had previously given* of the forces which must result when heat is communicated from a surface to a gas and the phenomena of the radiometer.

* Proc. Roy. Soc., 1874, p. 402.

Having found, what I had not at first perceived, that according to the kinetic theory the force resulting from the communication of heat to a gas must depend on the surface from which it is communicated being of limited extent, and must follow a law depending on some relation between the mean path of a molecule and the size of the surface, it appeared that by using vanes of comparatively small size the force should be perceived at comparatively greater pressures of gas (see Appendix, note 3).

On considering how this might be experimentally tested, it appeared that to obtain any result at measurable pressures the vanes would have to be very small indeed; too small almost to admit of experiment. And it was while thinking of some means to obviate this difficulty that I came to perceive that if the vanes were fixed, then instead of the movement of the vanes we should have the gas moving past the vanes—a sort of inverse phenomenon—and then instead of having small vanes, small spaces might be allowed for the gas to pass. Whence it was at once obvious that in porous plugs I should have the means of verifying these conclusions. I followed up the idea, and by a method which I devised of taking into account the forces, tangential and normal, arising from a varying condition of molecular gas, I was able to deduce what appears to me to be a complete theory of transpiration.

This theory appears to include all the results established by GRAHAM, as well as the known phenomena of the radiometer, which for the sake of shortness I shall call the *phenomena of impulsion.* I was also able definitely to deduce the results to be expected, both as regards thermal transpiration and the law of corresponding densities both for transpiration and impulsion.

Having made these deductions, I then commenced the experiments on transpiration, which so completely verified my theoretical deductions that I have been able to produce the theory in its original form, with some additions, but without any important modification.

Moreover, having succeeded (not without some trouble) in rendering apparent the effect of differences of temperature in causing gas to move through fine apertures, I recurred to the original problem, and by suspending fibres of silk and spider lines to act as vanes, I have now succeeded in directly verifying the conclusion that the pressure of gas at which the force in the radiometer becomes apparent varies inversely as the size of the vanes. With the fibre of silk I obtained repulsion at pressures of half an atmosphere.

The arrangement of the paper.

8. My object is to describe the reasoning by which I was led to undertake the experiments as well as the experiments themselves; but as the theory will be better understood after acquaintance with the facts, I have inverted the natural order and given the experiments first. And in order that the reader may not be at a disadvantage in reading the accounts of the experiments, I include here a somewhat fuller account of the results to be expected as deduced from the theory which is to follow.

The Laws established by the experiments.

9. Law I.—When gas exists at equal pressures on either side of a porous plate across which the temperature varies, the gas will transpire through the plate from the colder to the hotter side, with velocities depending on the absolute temperature and chemical nature of the gas, the relation between the density of the gas and the fineness of the pores, the thinness of the plug, and the difference of temperature on the two sides of the plate.

Law II.—In order to prevent transpiration through the plate, the pressure on the hotter side must be greater than the pressure on the colder side. This difference of pressure will depend on the chemical nature of the gas, the mean pressure of the gas, the absolute temperature, the relation between the size of the pores and the density of the gas, and the difference of temperature on the two sides of the plate, but not on the thickness of the plate.

Law III.—For the same plate and the same difference of temperature when the gas is sufficiently dense, the difference of pressure is approximately proportional to the inverse density, but as rarefaction proceeds this law gradually changes, the increase in the difference of pressure becomes less and less until that difference reaches a maximum and begins to diminish, then on further rarefaction this diminution increases until the difference of pressure becomes approximately proportional to the density of the gas.

Law IV.—After the rarefaction has reached that point at which the difference in pressure is nearly proportional to the density, then the difference in pressure will bear to the greatest pressure the ratio which the difference in the square roots of the absolute temperature bears to the square root of the greatest absolute temperature, or if A and B indicate the two sides of the plate,

$$\frac{P_A - P_B}{P_A} = \frac{\sqrt{\tau_A} - \sqrt{\tau_B}}{\sqrt{\tau_A}},$$

where p and τ represent respectively the pressure and the absolute temperature in the gas.

Respecting the results depending on the relation between the density of the gas and the fineness of the pores.

Law V.—Both in the case of thermal transpiration and of transpiration under pressure, similar results will respectively be obtained when the density of the gas bears a fixed relation to the diameters of the apertures in the plates.

Respecting the rate of transpiration arising from a difference of pressure on the two sides of the plate.

Law VI.—When gas exists at different pressures on the two sides of a plate, and the difference of pressure bears a fixed ratio to the pressure on either side; then for a certain plate and a certain gas the time of transpiration of equal volumes will, when the gas is sufficiently dense, be inversely proportional to the density; but as the rarefaction increases, the increase in the time of transpiration becomes less and less, until the time becomes constant.

Law VII.—When the rarefaction is so great that the time of transpiration of equal volumes of the same gas is constant, the times of transpiration of equal volumes of different gases will be proportional to the square root of the atomic weights of the gases.

Respecting the results of impulsion and the connexion between these results and the relation between the density of the gas and the size of the vanes.

Law VIII.—When the gas is sufficiently dense, then the impulsive force will be inversely proportional to the densities of the gas; but as the rarefaction proceeds the increase in the force becomes less and less until the rarefaction has reached a point depending on the size of the vanes (the larger the vanes the higher must be the rarefaction), after which the force begins to diminish, and ultimately diminishes with the density.

These laws were reduced to the form in which they have been stated in order to adapt them for experimental verification. Thus they do not represent the simplest nor yet the fullest form in which the properties of the gas can be expressed. This may be seen by reference to Sections X. and XII. which treat of the theory of these properties. There definite expressions will be found for the relations indefinitely indicated in Laws I. and II. These definite expressions are not introduced here, because they have not been definitely verified by experiment.

The definite relations expressed in Laws III., IV., V., VI., VII., and VIII., although derived from theoretical considerations, have all been to a greater or less extent verified by experiment—as far as the possible range of densities would admit—and in all cases the experimental results have within the limits of error corresponded well with the theoretical deductions.

SECTION II.—EXPERIMENTS RELATING TO THERMAL TRANSPIRATION.

10. In commencing these experiments it was impossible to form any estimate whatever of the magnitude of the results to be expected. The laws just stated only showed what would be the comparative value of the results under different circumstances; so that until a result had been found it was impossible to predict whether, with any particular plate, the result would be appreciable or not.

Thus it happened that although the experiments commenced on January 15, 1878, it was not until March that any definite results were obtained. This delay was chiefly owing to several very subtle sources of disturbance, the effect of which could only be distinguished from true results after a series of tests extending in each case over several days.

The material first used for the plates was WEDGEWOOD biscuit ware, $\frac{3}{16}$ths inch thick; and it was with this material after a long series of trials that connected results were first obtained. These results were very minute. With air at the pressure of the atmosphere, the greatest difference of pressure was ·1 of an inch (2·5 millims.) of mercury.

Having however once obtained this result, it was seen to follow from Law V., Art. 9, that greater results could be obtained with a finer plate. My idea was to try graphite, such as that used by GRAHAM; but in the meantime it occurred to me to try meerschaum, which proved to be a most convenient material, as it could be obtained in any sizes and readily cut into plates of any thickness.

With this material, first used on March 7, the later results were very striking; the difference of pressure amounting to ·25 of an inch with air at the pressure of the atmosphere, and to nearly an inch with hydrogen at the same pressure.

The description of the details of the earlier experiments, together with the various difficulties which were met with and the means employed to overcome them, would take too much room to admit of their being given at length. I shall therefore proceed at once to the description of the apparatus in its final form, and shall confine myself to noticing only such results as are important to the subject.

Description of the apparatus.

11. This consisted principally of an instrument which may be called a thermo diffusiometer.

This instrument, as shown in fig. 1a, Plate 47, consists essentially of two chambers separated by a plate of porous material, means being provided for keeping the chambers at constant but different temperatures for many hours at a time; also for measuring the pressure of gas in the chambers, for exhausting the chambers, and for bringing the chambers into direct communication when desired.

The chambers are formed by tin plates separated by rings of india-rubber, between which is held the porous plate. The external diameter of the rings is about $3\frac{1}{2}$ inches; and the internal diameter, the diameter of the chambers, is $1\frac{1}{2}$ inches. The thickness of the rings, the depth of the chamber, is about $\frac{3}{16}$ths of an inch. The porous plates are 2 inches in diameter, so that the edges are well covered by the rings of india-rubber which bound the chambers; and outside the plate is fitted another ring of india-rubber of the same thickness as the plate, so as to prevent any leak through the edges of the plate.

Outside the tin plates which form the walls of the chamber, other chambers are formed in the same manner by rings of india-rubber and tin plates. These second chambers afford the means of regulating the temperature, steam being continually passed through the one and cold water through the other. The chambers are made air-tight by means of pressure, which is brought to bear by means of a wooden press into which the rings and plates fit.

Fig. 1.

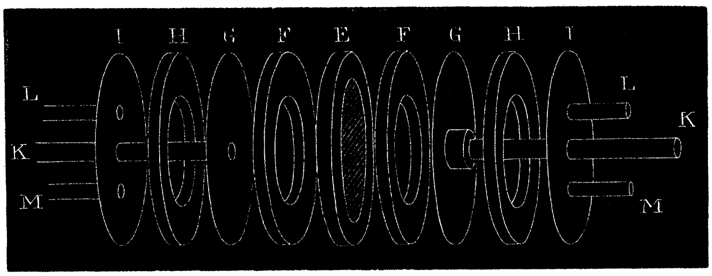


Figure 1 represents the plates and india-rubber rings somewhat separated.

E. Is the porous plate with the ring of india-rubber outside it.

FF. The rings which form the two chambers for gas on each side of the plate.

GG. The tin plates which close these chambers.

HH. The india-rubber rings which form the hot and cold chambers.

II. The tin plates which close these chambers.

KK. Tubes soldered to the tin plates GG to communicate with the chambers FF, and

LM, LM. Are tubes soldered to the tin plates II, to allow of the streams of steam or water through the chambers HH.

Fig. 2.

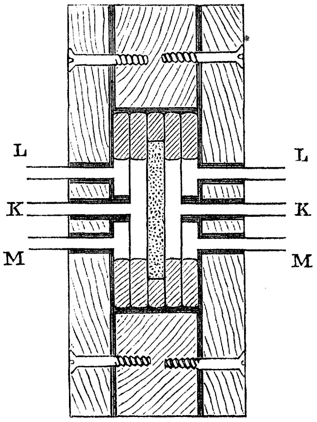


Figure 2 shows a section taken along the axis of the rings and plates, showing them in position, also the wooden press by which they are held together.

Conduction of heat.

12. The circumstance which principally led to the selection of this form of apparatus was the necessity of preventing, as far as possible, the conduction of heat from the hot to the cold side, through the material bounding the chambers. It will be seen that there is no metallic communication from the hot to the cold side, and that all the heat

which escapes across, besides what passes through the porous plate, must pass through something like half an inch of india-rubber, or through a considerably greater thickness of wood.

Communication with the chambers.

13. The communication with the gas chambers is effected by means of the tubes KK, the outward ends of which are fitted with three and four branches respectively.

By one of these branches the left chamber is connected with the open end of a mercurial vacuum gauge V or barometer tube, which measures the absolute pressure of this chamber.

Another branch from the left chamber, and a branch from the right, are respectively connected with the two ends of a siphon tube S containing mercury, which acts as a differential gauge for measuring the difference of pressure in the two chambers.

By means of the third branch from the left, and a second from the right, direct communication can be established between the chambers by turning a tap D.

The third and fourth branches on the right are used to establish communication with a mercurial pump and to admit dry gas.

These various connexions are shown in fig. 1a, Plate 47, which also shows the general arrangement of the apparatus.

The connexions between the metal and glass tubes are made with thick india-rubber tubing, $\frac{1}{8}$th inch bore and $\frac{3}{8}$th inch external diameter; and the two taps D and P shown in the sketch are both of glass.

The gauges.

14. *The vacuum gauge* is an ordinary barometer tube about 32 inches long and $\frac{1}{4}$ inch internal diameter, having its second limb sufficiently long to allow of the mercury standing level when the chambers were exhausted.

The differential gauge is of glass tube about $\frac{1}{8}$th inch internal diameter, it is altogether 30 inches long, so as to prevent the mercury being driven out of the tube by too great a difference of pressure.

Before the mercury was put into this tube it was wetted with sulphuric acid. A small quantity of this remained and covered the mercury on either side, by means of which sulphuric acid the free motion of the mercury was secured, so that differences of pressure as small as $\frac{1}{10000}$th of an inch of mercury caused it to move without the necessity of shaking.

Reading the gauges.

15. As far as the vacuum gauge was concerned, there was no point to be gained by extreme accuracy in reading the absolute pressure of gas in both chambers, so that a scale attached to the gauge was found to answer all purposes.

On the other hand the range of the experiments depended on the accuracy with which the differential gauge could be read. A special means of reading this gauge was devised. This consisted of a species of cathetometer almost close to the gauge, in which, instead of a telescope, a microscope with an inch object glass and a semi-disc in the focus of the eye-piece was used, the screw which moved the microscope had 50 threads to an inch, and the head had 200 divisions, so that one division corresponded to the $\frac{1}{10000}$th part of an inch. Owing to the high magnifying powers, the effect ot a motion of one division was visible, and several readings taken from the same position of the mercury agreed to within one division.

Testing the apparatus.

16. The complicated character of the apparatus and the number of joints rendered it extremely difficult to make it perfectly tight. When working at the pressure of the atmosphere this was of no great moment, but when working with rarefied gas it was necessary that it should be so tight that the leak might cause no appreciable disturbance.

At first india-rubber varnish was used to make the joints tight; but this did not answer, as the vapour from the varnish produced very considerable disturbance. After this the surfaces of the india-rubber were carefully washed, and then considerable pressure applied by wrapping wire on the tubes and screwing up the press. In this way, after a few days, the apparatus became what may be called perfectly tight. There was a slight leak or probably slight diffuson through the india-rubber, for after the experiments were concluded the apparatus was left full of hydrogen at the pressure of the atmosphere, and the tap communicating with the pump closed. It was then found that the pressure within the chambers steadily fell until it reached 9 inches of mercury. This point was reached after about one month. The pressure then began to rise, and in another month the gauge showed 12 inches. The entire volume of the chambers and tubes is only about 6 fluid ounces, so that it might well be imagined that the hydrogen had been absorbed by, or condensed on the india-rubber or the porous plate, but the fact that the pressure again rose seemed to imply that the hydrogen had escaped; but whether through the india-rubber or not it is impossible to say.

Such a leak, however, was entirely without effect on the results. In fact, a leak which admitted air at the rate of 1 inch of mercury in an hour into one chamber did not cause any appreciable alteration in the differential guage.

Drying the gas.

17. The presence of vapour in the gas was at first a source of great trouble. The tendency of porous plates to absorb moisture is so great, and the presence of vapour in the gas produces such a great disturbance even when the pressure of vapour is a long way below that at which it would condense on the cold surface, that for some time

this threatened to prevent any satisfactory result being obtained. At last, however, by having steam on both sides, and repeatedly exhausting and refilling with air that had been passed slowly through drying tubes, 40 inches long, containing first sulphuric and then anhydrous phosphoric acid—for which I am indebted to the kindness of Dr. ROSCOE—the effect of vapour was all but eliminated, and consistent results were obtained over several trials, even when *the sides of the steam and water were reversed.*

The differences of temperature.

18. The steam used for heating the apparatus was obtained by boiling water in a glass flask which held about a gallon, enough to last for twelve hours at a time. The flask was fitted with a water safety-valve; so that the pressure of steam could not exceed about 8 inches of water. The flask was placed about 6 feet from the instrument, so that the heat from the gas flame did not produce any material disturbance or materially affect the mercury in the gauges.

The cold water was direct from the main, and was found to be very constant in temperature, not varying throughout the experiments more than 23°—from 47° F. in February to 70° F. in July.

In this way the tin plates (GG, fig. 1) which bound the gas chambers were respectively maintained at temperatures differing by less than 1° F. from the temperature of the steam (212°) and that of the water.

The sides of the porous plate would not acquire the same temperatures as the steam and water, because the conduction through the porous plates would tend to equalise the temperature. Nor was there any means of ascertaining the exact temperatures other than by comparing the results obtained. But from these it appeared that there was considerable difference between the temperature of the surfaces of the porous plate and that of the opposite tin plate. A method of eliminating this difference has been found, and this will be explained with the results themselves.

The porous plates.

19. These, whether of biscuit ware, meerschaum, or stucco, were circular discs 2 inches (53·0 millims.) in diameter. The rings FF which formed the chamber had a diameter of $1\frac{1}{2}$ inches (38 millims.), and these limited the portion of the plate exposed to the passage of gas. The plates were of different thicknesses, the thinest being $\frac{1}{16}$th inch (1·5 millims.) and the thickest ·44 inch (14·2 millims.).

The results with air through porcelain plate No. 3 *and meerschaum Nos.* 1 *and* 2.

20. After numerous experiments, commencing on January 23, with plates Nos. 1, 2, and 3 of biscuit ware, the results of which, although there appeared to be a residual

difference of pressure, were very much disturbed, the first definite and consistent results were obtained with a porcelain plate No. 3, $\frac{1}{10}$th inch (2·5 millims.) thick, on February 22.

TABLE I.—Thermal transpiration of air by biscuit ware plate No. 3 (·1 inch or 2·5 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 47° F. or 8° C.

Mean pressure by vacuum gauge.		Difference of pressure by siphon gauge, February 22.		Ratio of mean pressure to difference of pressure.
inches.	millims.	inch.	millims.	
30	762	·1	2·54	300

This result was found to remain constant over a period of 8 hours, during which the steam and water were kept constantly flowing. It was also found to be the same whichever side of the diffusiometer was heated. During the experiment the tap bringing the hot and cold chambers into direct communication was frequently opened, and the differential gauge then indicated equal pressures. After each of these openings on the tap being again closed the same difference was re-established in a few seconds.

The next experiments were made with a somewhat thinner plate of meerschaum No. 1.

TABLE II.—Thermal transpiration of air by meerschaum plate No. 1 (·06 inch or 1·5 millims.). Temperature of steam, 212° F. or 100° C.; temperature of water, 47° F. or 8° C.

Mean pressure by vacuum gauge.		Difference of pressure by siphon gauge, March 12.		Ratio of mean pressure to difference of pressure.
inches.	millims.	inch.	millims.	
30	762	·08	2·03	350

As it seemed highly probable that the meerschaum plate was of finer texture than the porcelain plate previously tried, the fact that the difference of pressure with the meerschaum was not larger than with the porcelain was a matter of some surprise. There appeared, however, to be a possible cause for this in the thinness of the meerschaum. It was possible that there was some flaw in the plate, or more probably that the thinness of the plate allowed a considerable equalisation of temperature by the conduction of heat. It was therefore resolved to try a thicker plate of meerschaum, and a plate ·25 inch (6·3 millims.) was introduced in place of that previously tried.

TABLE III.—Thermal transpiration of air by meerschaum plate No. 2 (·25 inch or 6·3 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 47° F. or 8° C.

Mean pressure by vacuum gauge.		Difference of pressure by siphon gauge, March 15.		Ratio of mean pressure to difference of pressure.
inches.	millims.	inch.	millims.	
30·2	764·5	·25	6·096	126
12·9	327·6	·20	5·080	64
8·53	216·7	·17	4·318	50
3·70	94·0	·12	3·048	31
2·0	50·8	·08	2·032	25
0·88	12·35	·045	1·143	20
0·5	12·7	·035	0·889	13·6

Whether the fact that the thicker plate of meerschaum gave nearly three times the difference of either of the previous plates was due to the thicker plate maintaining a greater difference of temperature, or to some difference of texture in the thin plate, such as a flaw, has not been clearly determined, but it now appears probable that it was largely due to the first of these causes.

With this plate lower pressures were for the first time tried, and Table III. shows these differences falling with the pressure.

The ratio of the difference of pressure to the mean pressure, however, as is shown in the last column, increases as the pressure falls, and apparently is approximating to a constant value at lower pressures. This is according to Law III., Art. 9.

From Law IV., Art. 9, it appears that this ratio should, as the pressure fell, have approximated to the ratio which the difference of the square roots of the absolute temperature on the two sides of the plate bears to the square root of the temperature on the side at which the pressure was measured. Assuming $1 \div 13$ to be this ratio, it would appear that there must have been considerable differences of temperature between the surfaces of the meerschaum and the side of the plate; but it also appeared probable that with still lower pressures the ratio might have been considerably lower.

It would have been desirable to have carried the experiments to lower pressures, but at that time this was impossible as there was then no special means of reading the differential gauge; so that this had to be deferred until such a means was provided.

Hydrogen.

21. In the meantime other gases were tried. Owing to its lightness it was thought probable that hydrogen would at the higher pressure give a somewhat higher result than air. How much this might be the theory gave no certain indication, for it depended on qualities of the gas which had not been determined. But at the lower pressure, according to Law IV., the difference of pressure should approximate towards the same value relatively to the absolute pressure.

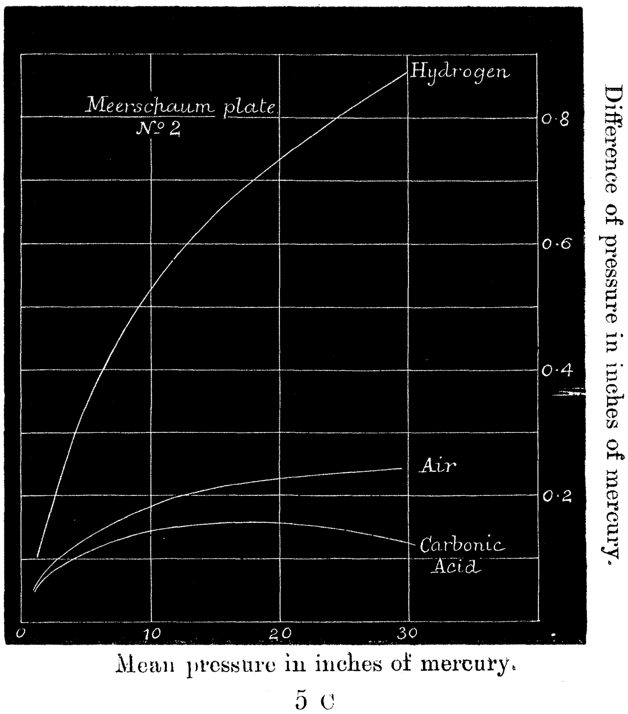
This was clearly a point which might be tested even though no very close approximation should be reached. Hydrogen was accordingly tried.

TABLE IV.—Thermal transpiration of hydrogen by meerschaum plate No. 2 (·25 inch or 6·3 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 47° F. or 8° C.

Mean pressure by vacuum gauge.		Difference of pressure by siphon gauge.		Ratio of mean pressure to difference of pressure.
inches.	millims.	inch.	millims.	
30·2	767	·88	23·37	32·4
13·0	330	·60	15·24	21
7·5	190·5	·44	11·18	17
4·25	107·9	·28	7·11	15
2·0	50·8	·15	3·81	13·3
1·0	25·4	·08	2·03	12·5
0·5	12·7	·036	0·91	13·7

This table shows that at the pressure of the atmosphere the difference with hydrogen was four times as great as it had been with air, and reached the very considerable figure of ·92 of an inch of mercury. This was much more than had been anticipated, although there was nothing in the theory to show that it should not exist. This great difference at the higher pressures only serves to bring out more forcibly the convergence according to Law IV. as the pressure falls. At pressures of 1 inch it will be seen that the differences for air and hydrogen are as 12·5 to 20, while if the results at ·5 inch could be trusted, the ratio is 13·7 to 13·6.

Fig. 3.



The convergence of these results is best seen in the accompanying diagram. The curves are drawn through points of which the pressures are abscissæ, and the differences of pressure (on a different scale) are the ordinates.

The maximum difference of pressure (*carbonic acid*).

22. The curves, fig. 3, show that the differences both for hydrogen and air appear to be tending, as the pressure rises, to a maximum value. This was exactly what was expected from Law III., Art. 9, and had the apparatus been capable of withstanding considerable pressures it would have been desirable to have raised the pressure until the maximum was passed. But it appeared that the same end might be more readily accomplished in other ways.

Owing to the great density and low coefficient of diffusion of carbonic acid, it seemed to be probable that with this gas the difference of pressure would reach a maximum at considerably lower pressures than either hydrogen or air. Carbonic acid was therefore tried.

TABLE V.—Thermal transpiration of carbonic acid by meerschaum plate No. 2 (·25 inch or 6·3 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 47° F. or 8° C.

Mean pressure by vacuum gauge.		Difference of pressure by siphon gauge, March 10.		Ratio of mean pressure to difference of pressure.
inches.	millims.	inch.	millims.	
30·1	764·5	·13	3·302	230
19·5	495·3	·16	4·064	122
14·25	361·8	·16	4·064	89
10·5	266·7	·15	3·810	70
8·0	203·2	·13	3·310	61
4·5	114·3	·11	2·794	40
2·0	50·8	·08	2·032	25
1·0	25·4	·05	1·270	20
0·5	12·7	·04	1·016	12

Table V. shows that with carbonic acid the maximum difference was at a pressure between 20 and 15 inches, the difference rising as the pressure fell from 30 inches to this point. After this point was passed the difference fell with the pressure.

The curve on fig. 3 which represents this table shows the point of maximum difference, and the figure also shows that as the pressures became small the curve for carbonic acid converges towards the curves for air and hydrogen.

These results for carbonic acid are perhaps sufficient to verify Law III. respecting the existence of a maximum. But they were obtained with considerable trouble, as the india-rubber tubing absorbed the carbonic acid very rapidly, and so caused considerable disturbance. For this reason carbonic acid was not again used.

Stucco plate No. 1.

23. As it appeared from Law V., Art. 9, that any increase in the coarseness of the plate should reduce the pressure at which the difference should be a maximum for each gas, a plate of stucco was tried with this object.

It was clear that the differences would be much smaller with the stucco than with the meerschaum. Therefore this plate was not tried until the differential gauge had been furnished with the cathetometer to read to $\frac{1}{10000}$th of an inch ($\frac{1}{400}$th of a millim.).

Final experiments.

A series of experiments, commencing with stucco plate No. 1, but continued with meerschaum plate No. 3 and stucco No. 2, were commenced on May 6, and repeated in July. To give all the observations made in this series of experiments would occupy too much space, therefore a selection has been made, those results being chosen which appeared to be least subject to disturbance. However, the results all agree so well that there was but little choice, and it was clearly unnecessary to resort to the usual method of taking mean values. Such differences as do exist are sufficiently accounted for by the small differences in the temperature of the water, which was several degrees higher in July than in May.

TABLE VI.—Thermal transpiration of air by stucco plate No. 1 (·25 inch or 6·3 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 65° F. or 18°·4 C.

Mean pressure by vacuum gauge.		Difference of pressure by siphon gauge, July 11.		Ratio of mean pressure to difference of pressure.	Log of mean pressure.	Log of difference of pressure.
inches.	millims.	inch.	millims.			
29·80	756·9	·0220	·559	1360	2·474—1	2·342—4
28·50	723·9	·0225	·571	,,	2·455	2·352
25·85	656·6	·0235	·597	..	2·412	2·371
23·40	594·4	·0250	·635	903	2·369	2·397
22·20	563·9	·0266	·675	835	2·346	2·424
17·40	443·9	·0294	·746	..	2·240	2·468
15·40	391·2	·0336	·813	..	2·187	2·526
13·60	345·4	·0342	·868	..	2·133	2·534
12·25	311·1	·0348	·884	..	2·066	2·541
11·35	288·3	,,	,,	326	2·053	2·541
10·00	254·0	·0366	·929	..	2·000	2·563
9·00	228·6	·0380	·965	..	1·954	2·579
7·50	190·5	,,	,,	200	1·875	2·579
6·75	171·4	·0376	·955	..	1·630	2·575
6·00	152·4	,,	·955	..	1·778	2·575
5·15	130·8	·0362	·917	142	1·711	2·559
4·35	110·5	·0354	·899	120	1·638	2·549
3·50	88·9	·0306	·828	107	1·544	2·513
2·90	73·7	·0314	·797	92	1·462	2·496
2·35	59·7	·0290	·736	81	1·370	2·462
2·25	57·15	·0284	·721	79	1·350	2·453
1·25	31·75	·0230	·584	54	1·097	2·361
0·60	15·24	·0149	·378	42	1·778	2·258
0·25	6·35	·0080	·203	31	1·398	1·903
0·15	2·66	·0066	·167	23	1·176	1·819

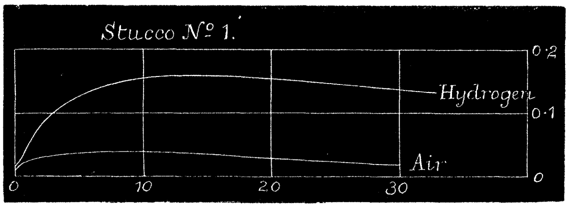
TABLE VII.—Thermal transpiration of hydrogen by stucco plate No. 1 (·25 inch or 6·3 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 63° F. or 17° C.

Mean pressure by vacuum gauge.		Difference of pressure by siphon gauge.			Ratio of mean pressure to difference of pressure.	Log of mean pressure.	Log of difference of pressure.
		May 6.	July 11.				
inches.	millims.	inch.	inch.	millims.			
33·00	858·0	..	·1340	3·404	252	2·518—1	3·127—4
31·00	787·4	..	·1366	3·470	227	2·491	3·135
29·00	736·6	..	·1396	3·546	207	2·462	3·145
28·50	723·9	·1408	..	3·576	203	2·454	3·149
,,	,,	..	·1400	3·556	,,	,,	3·147
27·00	685·8	..	·1436	3·647	188	2·431	3·157
25·30	642·6	..	·1446	3·672	174	2·403	3·160
23·75	603·2	..	·1460	3·708	162	2·375	3·164
22·00	558·8	..	·1490	3·784	147	2·342	3·173
20·00	508·0	..	·1530	3·886	130	2·301	3·185
19·00	482·6	..	·1540	3·912	123	2·279	3·187
18·00	457·2	..	,,	3·912	116	2·255	3·187
16·70	424·1	..	·1542	3·917	109	2·222	3·188
16·00	406·4	·1532	..	3·891	104	2·204	3·185
15·80	401·3	..	·1532	,,	103	2·199	3·185
14·90	378·4	..	·1538	3·906	94	2·178	3·187
13·35	339·0	..	·1536	3·901	87	2·125	3·186
12·50	317·5	..	·1534	3·896	81	2·096	3·186
11·55	393·4	..	·1512	3·840	76	2·062	3·179
9·80	248·9	..	·1506	3·825	65	1·991	3·178
9·50	239·7	·1512	..	3·840	62·5	1·977	3·170
9·00	228·6	..	·1480	3·759	60·8	1·954	3·175
8·00	203·2	..	·1470	3·734	55	1·903	3·167
6·00	152·4	..	·1320	3·353	45	1·778	3·120
3·25	82·5	..	·1046	2·637	31	1·511	3·019
3·2	81·3	..	·1020	2·590	,,	1·505	3·008
2·0	50·8	..	·0760	1·930	25	1·301	2·880
1·8	45·72	·0760	..	,,	23·5	1·255	2·880
1·15	29·21	..	·0500	1·270	23	1·176	2·698
0·7	17·78	..	·0330	·838	21	0·845	2·518
0·6	15·24	·0280	..	·711	,,	0·778	2·447
0·4	10·16	..	·0190	·482	,,	0·602	2·278
0·3	7·62	·0158	..	·401	19	0·477	2·198

With the stucco plate the greatest differences of pressure, both in the case of air and that of hydrogen, are small, something less than one-fourth the differences previously found in the case of the meerschaum plate No. 2; but then with the stucco the points of maximum difference are well below the pressure of the atmosphere.

The difference of pressure between the observations is so small, and the agreement of the observations so great, that by merely joining the points plotted to represent the observations, very fair curves are formed.

Fig. 4.



These curves bring out in a marked manner the agreement of the results with Law III., Art. 9.

With air the difference rises from ·02 of an inch (·508 millim.), at a pressure of 30 inches, to ·0380 of an inch or (·965 millim.), at a pressure of 7·5 inches, which is the maximum.

With hydrogen the difference also rises as the pressure falls from 30, but the rise is not so great and the maximum is reached at 16 inches.

After passing the maximum the curves both fall, and in falling obviously converge.

This is all exactly in accordance with what was expected.

Corresponding pressures (*stucco* 1, *meerschaum* 2).

24. Law V. shows that there should be correspondence between certain portions of the curve for stucco and those for meerschaum, although the corresponding points would not be at the same pressures.

Assuming the temperatures to be the same, the corresponding points would be those for which the ratio of the mean pressure to the difference of pressure were the same. Which points may at once be found by comparing the figures in the columns showing this ratio in Tables III. and IV., with the same columns in Tables VI. and VII. respectively.

Before making such a comparison, however, it is necessary to introduce certain small corrections for the difference in the temperature of the water in the two experiments; this, as will be subsequently explained, will be equivalent to diminishing the difference in the Tables III. and IV. in the ratio 7 to 8.

Then we find that the pressures at which the ratios are the same in Tables III. and VI. are approximately as 6 to 1, taking only the higher pressures, while the Tables IV. and VII. give the ratio 6·7 to 1.

The results of this comparison, although not strictly consistent, indicate that there is a correspondence, the points on the curves for meerschaum corresponding with points on the curves for stucco, for which the pressures are about $\frac{1}{6}$ for air, and $\frac{1}{6\cdot 7}$ for hydrogen.

It was clear, however, that the number of observations with the meerschaum plate was not sufficient to allow of a very close comparison with the curve for stucco, for the accuracy with which the differences had been read without the cathetometer was not sufficient to allow of any use being made of the lower pressures.

Meerschaum plate No. 3.

25. A fresh meerschaum plate, ·44 inch thick, was therefore tried, another diffusiometer, exactly similar to the original one, being constructed for the purpose.

TABLE VIII.—Thermal transpiration of air by meerschaum plate No. 3 (·44 inch or 11·2 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 63° F. or 17° C.

Mean pressure by vacuum gauge.		Difference of pressure by siphon gauge.			Ratio of mean pressure to difference of pressure.	Log of mean pressure.	Log of difference of pressure.
		May 11.	May 14.				
inches.	millims.	inch.	inch.	millims.			
31·00	787·4	·2200	. .	5·588	141	2·49−1	2·342−3
29·50	749·3	. .	·2140	5·436	138	2·47	2·330
28·50	723·9	·2160	. .	5·486	132	2·45	2·334
27·50	698·5	. .	·2126	5·400	129	2·44	2·327
24·50	622·3	. .	·2100	5·334	116	2·37	2·322
23·00	584·2	·2130	. .	5·410	108	2·36	2·328
21·50	546·1	. .	·2054	5·217	104	2·33	2·312
20·00	508·0	·2120	. .	5·385	94	2·30	2·326
19·50	495·3	. .	·1970	5·003	99	2·29	2·294
18·00	457·2	·2100	. .	5·334	85	2·25	2·322
17·00	431·8	. .	·1890	4·800	90	2·23	2·276
12·50	317·5	·1730	. .	4·394	72	2·09	2·238
11·50	292·1	. .	·1630	4·140	70	2·06	2·212
8·25	209·5	·1446	. .	3·672	57	1·92	2·160
7·80	198·1	. .	·1336	3·393	59	1·89	2·105
5·20	133·3	·1184	. .	3·007	44	1·72	2·073
4·70	118·4	. .	·1050	2·667	,,	1·672	2·021
3·40	86·4	·0904	. .	2·290	37	1·531	1·954
3·10	78·7	. .	·0806	2·047	38	1·491	1·906
2·10	53·3	·0710	. .	1·803	29	1·322	1·851
2·00	50·8	. .	·0630	1·604	32	1·301	1·799
1·40	35·6	·0510	. .	1·294	27	1·146	1·707
1·32	33·5	. .	·0486	1·234	,,	1·120	1·687
1·10	28·0	·0394	. .	1·000	28	1·041	1·595
0·83	21·06	. .	·0380	0·965	22	0·919	1·580
0·65	16·51	·0304	. .	0·762	21	0·812	1·482
0·52	13·21	. .	·0290	0·736	18	0·716	1·462
0·40	10·16	·0250	. .	0·635	16	0·544	1·392
0·39	9·91	. .	·0220	0·559	17	0·531	1·342
0·28	7·11	·0192	. .	0·488	14	0·361	1·283
,,	,,	. .	·0180	0·457	14	0·361	1·255
0·20	5·08	·0168	. .	0·427	12	0·301	1·225
0·19	4·82	. .	·0146	0·371	12	0·278	1·164
0·15	3·81	·0154	. .	0·391	10	0·176	1·187

TABLE IX.—Thermal transpiration of hydrogen by meerschaum plate No. 3 (·44 inch or 11·2 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 63° F. or 17° C., May 15 and 18; temperature of water, 65° F. or 18° C., July 10.

Mean pressure by vacuum gauge.		Difference of pressure by siphon gauge.				Ratio of mean pressure to difference of pressure.	Log of mean pressure.	Log of difference of pressure.
		May 15.	May 18.	July 10.				
inches.	millims.	inch.	inch.	inch.	millims.			
35·00	889·0	. .	. .	·7940	20·17	44	2·544—1	2·900—3
34·00	863·6	. .	·7930	. .	20·14	43	2·531	2·899
32·00	812·8	·7930	. .	. .	20·14	40	2·505	2·899
30·00	762·0	. .	·7670	. .	19·48	39	2·477	2·885
29·50	749·3	·7760	. .	. .	19·71	38	2·470	2·889
”	”	. .	. .	·7776	19·75	37	”	2·890
27·50	698·5	·7600	. .	. .	19·30	36	2·439	2·880
22·00	558·8	·6914	. .	. .	17·56	32	2·342	2·840
18·50	469·9	. .	·6250	. .	15·87	29·5	2·267	2·795
18·00	457·2	·5710	. .	. .	14·50	31	2·255	2·757
”	”	. .	. .	·5960	15·14	30	2·255	2·775
12·00	304·8	. .	. .	·4800	12·19	25	2·070	2·681
11·40	289·6	. .	·4626	. .	11·75	24·6	2·056	2·664
10·50	266·7	·4156	. .	. .	10·56	25	2·021	2·618
7·70	195·6	. .	. .	·3594	9·13	21	1·886	2·555
7·60	193·0	. .	·3160	. .	8·03	24	1·880	2·500
6·95	176·5	·3046	. .	. .	7·74	23	1·842	2·484
4·75	120·6	. .	·2206	. .	5·60	23	1·676	2·343
”	”	. .	. .	·2584	6·57	18	”	2·412
4·50	114·3	·2120	. .	. .	5·38	21	1·653	2·326
3·00	76·2	. .	·1568	. .	3·98	19	1·477	2·200
”	”	. .	. .	·1784	4·53	17	”	2·251
2·60	66·0	·1420	. .	. .	3·60	18	1·414	2·152
1·95	49·6	. .	. .	·1204	3·06	16	1·290	2·080
1·70	43·2	·1063	. .	. .	2·70	16	1·230	2·026
1·25	31·8	. .	. .	·0784	1·991	16	1·096	1·894
1·10	27·95	. .	·0630	. .	1·600	17	1·041	1·799
1·00	25·40	·0660	. .	. .	1·676	15	1·000	1·819
0·70	17·78	. .	. .	·0380	0·965	18	0·845	1·580
0·65	16·51	. .	·0325	. .	0·825	20	0·813	1·511
0·60	15·26	·0380	. .	. .	0·965	15	0·778	1·500
0·35	8·88	. .	·0250	. .	0·635	14	0·544	1·297
0·32	8·13	·0200	. .	. .	0·580	16	0·505	1·301
0·175	. .	·0150	. .	. .	0·381	12	0·243	1·176

Although this plate was so much thicker than meerschaum plate No. 2, the results were no greater. They appear rather less, but this was owing to the somewhat higher temperature of the water, which would reduce the results in Table IV. in the ratio 8 to 9, and when this correction is applied the agreement is very close.

Effect of the thickness of the plate.

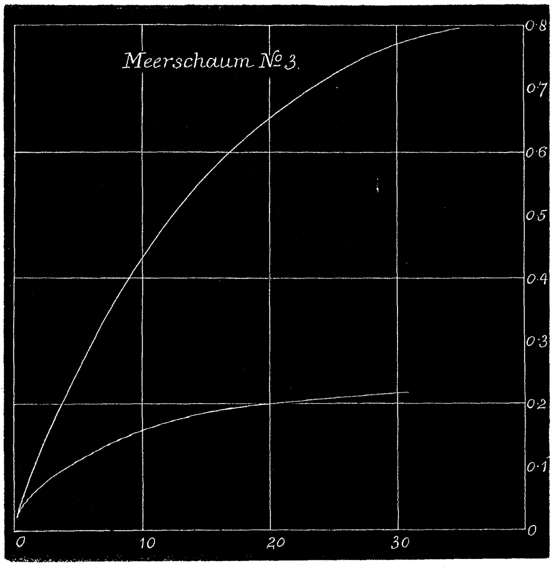
26. It had been expected, however, that the extra thickness of the plate No. 3 would have caused it to give somewhat higher results, and its not doing so seemed to

imply that the plates were so thick that the conduction of heat through the plate produced no appreciable effect on the temperature of the surfaces of the meerschaum. It appeared, however, from subsequent experiments that in all probability there was a small difference in the two instruments. The original instrument, that in which the experiments on plate No. 2 were made, had been used a great deal, and the surfaces of the tin plates which were opposite to the meerschaum had lost all their polish and become black, while in the second instrument the plates were new and bright. It might, therefore, be expected that the old plates would radiate more heat than the bright plates, and so better maintain the difference of temperature, and besides this the india-rubber rings in the new instrument were somewhat thicker than those in the old one, and so the space between the plates and the surface of the meerschaum was greater than in the old instrument. It appears, therefore, that these causes may have neutralised the increase in the difference of temperature that would otherwise have resulted from the extra thickness of the plate. And it will be seen that this conclusion was confirmed when on introducing a new stucco plate into the old instrument new tin plates and thicker rings were also introduced.

Infusion of air.

The curves, fig. 5, show the degree of regularity attained in these experiments. Such discrepancies as there are, are apparently owing to the absorption and exhalation of the gas by the india-rubber and possibly by the plate itself, for these discrepancies only occur at the lower pressures.

Fig. 5.



In the case of hydrogen the greatest care was taken to get the gas pure; but it is not to be supposed that as the gas was pumped out the residual gas would maintain a high degree of purity, for the gases given off by the india-rubber and the air which diffused through it would gradually replace the hydrogen.

Corresponding pressures with stucco No. 1 *and meerschaum No.* 3.

27. Comparing the ratio columns in Tables VIII. and IX. with the corresponding columns in Tables VI. and VII. respectively, the corresponding pressures are found to be as shown in Tables X. and XI.

TABLE X.—Showing the pressures of air for which the ratio of the difference of pressure to the mean pressure is the same for stucco No. 1 and meerschaum No. 3.

Ratio of mean pressure to difference of pressure.	Meerschaum No. 3. Pressure.	Stucco No. 1. Pressure.	Ratio of corresponding pressures.
	inches.	inches.	
141	31·0	5·1	6·08
138	29·5	5·0	5·90
132	28·5	4·75	6·00
129	27·5	4·6	6·00
116	24·5	4·0	6·1
108	23·0	3·7	6·2
104	21·50	3·5	6·1
94	20·00	3·0	6·6
99	19·50	3·25	6·0
85	18·00	2·6	6·9
90	17·00	2·8	6·05
72	12·5	1·9	6·5
70	11·5	1·8	6·3
57	8·25	1·15	7·0
59	7·8	1·25	6·2
44	5·25	0·5	10·5
44	4·70	0·5	9·0
37	3·40	0·35	9·9
38	3·10	0·35	9·0
29	2·10	0·24	8·5
32	2·00	0·301	6·6
27	1·40	0·20	7·0
27	1·32	0·20	6·0

TABLE XI.—Showing the pressures of hydrogen at which the ratio of the difference of pressure to the mean pressure is the same for meerschaum No. 3 and stucco No. 1.

Ratio of mean pressure to difference of pressure.	Corresponding pressures.		Ratio of corresponding pressures.
	Meerschaum No. 3.	Stucco No. 1.	
	inches.	inches.	
44	35	5·8	6·0
43	34	5·5	6·2
40	32	5·0	6·4
39	30	4·8	6·2
38	29·5	4·6	6·4
37	29·5	4·4	6·7
36	27	4·2	6·4
32	22	3·4	6·4
29·5	18·5	2·9	6·3
31	18	3·2	5·6
30	18	3·0	6·0
25	12	2·0	6·0
24·6	11·40	1·9	6·0
25	11·50	2·0	5·25
21	7·70	8·0	9·7
24	7·60	1·7	4·5

In Tables X. and XI. the first columns are the ratios taken direct from Tables VIII. and IX., the second columns are the pressures also taken direct from Tables VIII. and IX.

In order to find the pressures with the stucco plate which would yield exactly the same ratios (difference of pressure to mean pressure) as those in the table, the numbers in the ratio columns of Tables VI. and VII. were plotted, the mean pressures being taken as abscissæ. The points were joined so as to form curves, and then finding points on the curve whose ordinates corresponded to a particular number in the first column, the abscissæ gave the numbers required for the third column in Tables X. and XI. In this way the numbers in the third column are rather more uniform than they would have been had they been the results of actual observation.

Tables X. and XI. show that within the limits of accuracy of the experiments the pressures in the stucco correspond with pressures in the meerschaum six times as great. This is exactly according to Law V., Art. 7, from which it appears that the numerical relation between the corresponding pressures is the relation between the diameters of the interstices of the meerschaum and stucco plates. This fact also is confirmed, for not only does it appear that the ratio is independent of the mean density of the gas, but it is the same for hydrogen as it is for air, showing that the relation depends only on the nature of the plates.

Logarithmic homologues of the curves in figs. 4 *and* 5.

28. It appeared, however, that as a method of obtaining the corresponding pressures the comparison of the ratios was not entirely satisfactory, for it involved the assumption that the ratio of corresponding differences of pressure should be exactly the same as the ratio of corresponding mean pressures; whereas this would only be the case if the differences of temperature were exactly the same for both plates. It seemed desirable therefore to find a means of comparing the curves for the two plates on the assumption that the corresponding abscissæ might bear one ratio and the corresponding ordinates another, or if 1 and 2 are corresponding points, $x_2=ax_1$ while $y_2=by_1$.

A graphic method of doing this simply and perfectly was found by comparing not the curves themselves, but what may be called their *logarithmic homologues.*

Instead of plotting, as in figs. 3 and 4, the mean pressures and differences of pressure as the abscissæ and ordinates of the points on the curve, the logarithms of these quantities are plotted. Thus, $x'_2= \log x_1$, $y'_1= \log y_1$, where x_1y_1 may be taken to be a point on any one of the curves already plotted, and $x'_1y'_1$ the corresponding point on the logarithmic homologue. It is thus seen that if for two curves (1) and (2), $x_2=ax_1$ and $y_2=by_1$, then $x'_2=x'_1+\log a$ and $y'_2=y'_1+\log b$; or the logarithmic homologues will all be similar curves but differently placed with regard to the axes, such that the one curve may be brought into coincidence with the other by a shift of which the coordinates are $\log a \log b$.

Fig. 6.

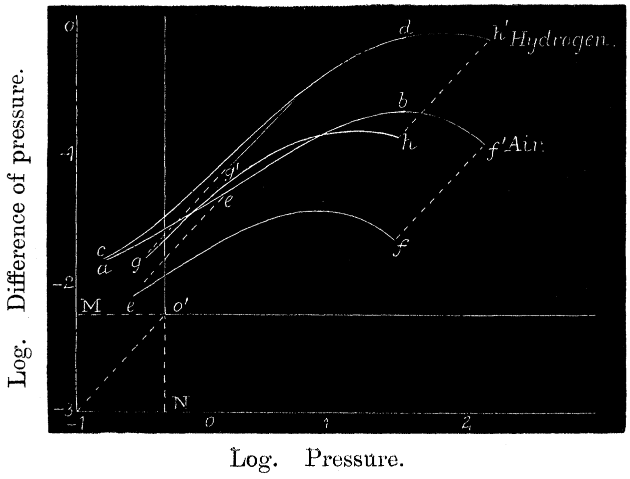


Fig. 6 shows the logarithmic homologues of the curves for stucco No. 1 and meerschaum No. 3 both for hydrogen and air. By tracing the log curves for stucco No. 1, together with the axes, on a piece of tracing paper, and then moving the tracing (so that the axes remain parallel to their original direction) until the traced curves fit on to the curves for meerschaum No. 3, it is found that the fit is perfect, a portion of the traced curve $e'f'$ (stucco) coinciding with a portion of $a\ b$, while at the same time a portion of the traced curve $g' h'$ coincides with a portion of $c\ d$. The effect of the superposition is shown in the figure, $e' b$ and $g' d$ being the portions of the curves which overlap. O′ is the new position of O.

It will at once be seen that O′ M is the logarithm of the ratio of corresponding abscissæ, while O′ N is the logarithm of the corresponding ordinates.

In this particular case

$$O' N = \cdot 7 = \log.\ 5$$
$$O' M = \cdot 77 = \log.\ 5\cdot 9.$$

These numbers differ somewhat from those given by Tables X. and XI., and the difference is very suggestive. The absolute agreement of the curves shows that the difference is not owing to experimental inaccuracy, and it will be seen on comparing the results next given that the difference (5 and 5·9) is owing to a difference in the temperature in the two instruments. If the temperatures had been the same we should have had the same ratio for the corresponding ordinates as for the abscissæ; but a difference in the temperature would alter all the ordinates in a certain ratio without affecting the abscissæ.

The difference $O' N - O' M = \cdot 07 = \log 1\cdot 175$ gives the ratio in which the differences of pressure are affected by a difference in temperature. This, according to the law that the results are proportional to the square roots of the differences of temperature, would be equivalent to a difference of 21° in the temperature of the water. This difference did not exist, hence there must have been a difference, owing to the greater thickness or to the different nature of the meerschaum plate.

The size of the woodcut does not admit the points indicating the actual experiments being shown, but these are shown in the larger figures, Plates 48 and 49.

Stucco plate No. 2.

29. These facts will be better understood after examining the experiments on a second stucco plate. The trial of this plate was owing to an accident to the diffusiometer containing stucco plate No. 1. The diffusiometer was thereupon refitted with another plate similar to No. 1; but the old tin plates were replaced by new bright ones, and the new india-rubber rings were somewhat thicker than the old ones.

TABLE XII.—Thermal transpiration of air by stucco plate No. 2 (·25 inch or 6·35 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 70° F. or 21° C.

Mean pressure by vacuum gauge.		Difference of pressure by siphon gauge.			Ratio of mean pressure to difference of pressure.	Log of mean pressure.	Log of difference of pressure.
		July 17.	July 18.				
inches.	millims.	inch.	inch.	millim.			
30·25	768·3	·0160	. .	·406	1892	2·480—1	1·204—3
30·05	763·3	. .	·0162	·411	1855	2·477	1·209
28·05	712·4	·0166	. .	·422	1710	2·448	1·220
27·25	692·1	. .	·0170	·432	1600	2·435	1·230
25·85	656·6	·0176	. .	·447	1470	2·412	1·245
24·90	632·4	. .	·0180	·457	1383	2·396	1·255
23·15	588	·0196	. .	·498	1181	2·364	1·292
22·05	560	. .	·0194	·492	1137	2·343	1·287
20·30	515·6	·0208	. .	·528	976	2·307	1·318
19·20	487·4	. .	·0204	·518	946	2·283	1·309
18·00	457·2	·0230	. .	·584	784	2·255	1·361
16·10	408·94	·0240	. .	·610	670	2·207	1·380
15·8	401·32	. .	·0230	·584	680	2·199	1·361
14·0	355·6	·0254	. .	·645	551	2·146	1·404
13·80	350·52	. .	·0244	·620	565	2·140	1·387
12·45	316·2	·0266	. .	·676	453	2·095	1·425
11·85	301·00	. .	·0256	·650	462	2·073	1·408
10·82	274·8	·0276	. .	·701	391	2·034	1·440
10·05	255·2	. .	·0262	·660	383	2·002	1·418
9·80	248·9	·0282	. .	·716	348	1·991	1·450
9·10	231·1	. .	·0272	·691	334	1·959	1·434
8·75	222·1	·0284	. .	·721	308	1·942	1·453
8·10	205·7	. .	·0280	·711	290	1·908	1·447
7·65	194·3	·0290	. .	·736	264	1·883	1·462
7·15	181·6	. .	·0284	·721	252	1·853	1·453
6·72	170·7	·0294	. .	·746	229	1·817	1·468
6·20	157·5	. .	·0288	·731	215	1·791	1·459
5·50	139·7	·0290	. .	·746	190	1·740	1·462
5·25	133·3	. .	·0286	·726	183	1·720	1·456
4·40	111·7	·0280	. .	·711	157	1·643	1·447
4·30	109·2	. .	·0276	·701	156	1·633	1·440
3·40	86·4	·0266	. .	·676	128	1·531	1·422
3·35	85·1	. .	·0264	·671	127	1·525	1·421
2·70	68·6	·0242	. .	·615	112	1·431	1·381
2·40	60·96	. .	·0226	·574	106	1·380	1·354
2·00	50·8	·0214	. .	·543	93	1·301	1·330
1·45	36·8	. .	·0182	·462	80	1·161	1·266
1·22	31·00	·0176	. .	·447	70	1 086	1·245
·80	20·82	·0138	. .	·350	58	0·903	1·139
·50	12·70	. .	·0108	·274	48	0·699	1·033
·38	9·65	. .	·0088	·223	42	0·580	0·944
·225	5·71	·0050	. .	·127	45	0·352	0·699

TABLE XIII.—Thermal transpiration of hydrogen by stucco plate No. 2 (·25 inch or 6·35 millims. thick). Temperature of steam, 212° F. or 100° C.; temperature of water, 70° F. or 21° C.

Mean pressure by vacuum gauge.		Difference of pressure by siphon gauge.			Ratio of mean pressure to difference of pressure.	Log of mean pressure.	Log of difference of pressure.
		July 19.	July 20.				
inches.	millims.	inch.	inch.	millims.			
31·00	787·40	. .	·1072	2·723	290	2·491—1	2·030—3
30·55	775·50	·1080	. .	2·743	283	2·484	2·033
29·60	751·60	. .	·1084	2·753	273	2·456	2·035
28·10	713·70	. .	·1102	2·799	255	2·449	2·042
26·70	677·90	. .	·1122	2·850	237	2·426	2·050
25·50	647·50	. .	·1132	2·875	225	2·406	2·054
25·25	641·00	·1130	. .	2·870	223	2·402	2·053
24·15	613·40	. .	·1152	2·916	209	2·383	2·061
22·40	568·90	·1180	. .	2·997	190	2·350	2·072
22·05	560·00	. .	·1176	2·977	188	2·343	2·070
21·10	535·90	. .	·1182	3·002	177	2·324	2·072
20·15	510·50	. .	·1186	3·012	170	2·304	2·074
20·00	508·00	·1192	. .	3·027	168	2·301	2·076
19·20	487·60	. .	·1190	3·023	160	2·283	2·075
17·15	435·60	. .	·1204	3·058	142	2·234	2·080
16·20	411·40	. .	·1208	3·068	134	2·209	2·082
16·00	406·40	·1214	. .	3·083	130	2·204	2·084
15·30	388 60	. .	·1214	,,	126	2·185	2·084
14·60	370·80	. .	·1220	3·098	119	2·164	2·086
14·55	369·50	·1220	. .	,,	,,	2·163	2·086
13·95	354·30	. .	·1220	,,	114	2·144	2·086
13·20	335·20	·1212	. .	3·078	108	2·120	2·083
12·35	317·70	. .	·1216	3·088	101	2·091	2·085
11·95	281·80	·1200	·1200	3·048	100	2·077	2·070
10·70	271·80	·1198	. .	3·043	89	2·029	2·087
9·60	243·80	·1176	. .	2·987	80	1·982	2·070
8·65	219·70	·1146	. .	2·910	75	1·937	2·059
7·75	196·80	·1120	. .	2·844	69	1·889	2·049
6·30	160·00	·1064	. .	2·702	60	1·799	2·027
5·75	146·00	. .	·1000	2·540	56	1·759	2·000
5·10	129·50	·0976	. .	2·479	52	1·700	1·989
3·65	92·70	. .	·0854	2·169	42	1·562	1·931
3·40	86·30	. .	·0860	2·184	40	1·531	1·934
2·50	63·50	·0704	. .	1·788	35	1·398	1·847
1·60	40·00	. .	·0524	1·331	30	1·204	1·719
1·10	27·90	. .	·0420	1·066	26	1·041	1·623
·35	8·88	·0170	. .	·431	20	0·544	1·230

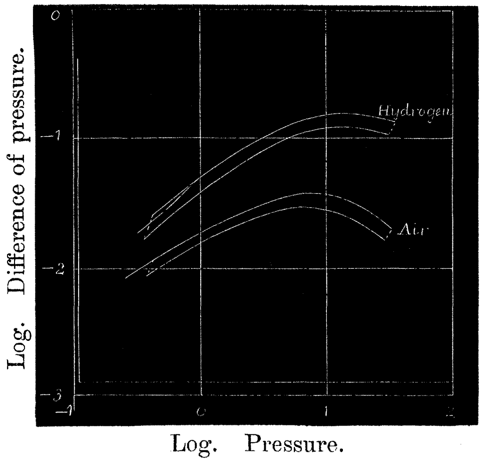
In making the experiments contained in Tables XII. and XIII., there was a slight change from the former plan, which had been to begin at the higher pressures and thence proceed by successive exhaustions to the lower pressures. This time one series of experiments was made as before, and another in the inverse manner—the diffusiometer being exhausted to commence with and the air or hydrogen being allowed to enter between the observations. Both series are given in the tables and are seen to agree very closely.

In the case of hydrogen it was found as before, that although not great, there was still a greater tendency to irregularity than with air, and as this was evidently due to diffusion through the india-rubber during the very considerable time (4 or 5 hours) which elapsed before the lower pressures were reached, several independent experiments were made. The diffusiometer being filled with pure hydrogen, was exhausted at once down to the particular low pressure at which the reading was taken, so as not to allow time for diffusion.

Differences of temperature brought to light by the log. curves.

30. The results with stucco plate No. 2 are smaller than with No. 1. At first sight it was thought that this difference was entirely owing to No. 2 being somewhat coarser than No. 1, but when the logarithmic homologues of the curves for this plate came to be compared with those for No. 1 and meerschaum No. 3, after the manner described in Art. 28, it became apparent that the difference in the results with plates No. 1 and 2 (stucco) was due to two causes. Some of it was due, as had been supposed, to the greater coarseness of No. 2, but a large part could only be explained on the assumption that from some cause or another the difference of temperature with No. 2 was less than with No. 1.

Fig. 7.



In fig. 7 it is seen that in order to bring the log. curves for stucco No. 2 into coincidence with the curves for stucco No. 1, it was necessary to increase the abscissæ of the former by ·048—log. 1·117 : while the ordinates had to be increased by ·112. The difference in the abscissæ, as shown in Art. 28, represents the difference due to the coarseness of the plate; thus the openings in No. 2 are 1·117 times as broad as the openings in No. 1. And the difference between the differences of the ordinates and the abscissæ =·112—·048=·064=log. 1·16 is the logarithm of the effect of a difference of temperature, and to produce this effect the temperature of the water would have had to be lowered 15°. There was some difference, from 5° to 7°, leaving from 8° to 10° to be expressed as due to the bright tin plates and thicker rings.

Comparison of the logarithmic homologues.

31. In Plates 48 and 49 the curves for meerschaum are drawn in the same position with reference to the axis, O M, O N. But while in Plate 48 the curves for stucco No. 1 and No. 2 are shown in the position as plotted from the columns of logarithms in the Tables VI., VII., XII. and XIII., in Plate 49, these curves have all been shifted until they coincide with the curves for meerschaum No. 3, in each case the two curves for air and hydrogen being shifted together. The axes are also shown as shifted with each pair of curves. The fitting of these curves is very remarkable; nor is it only the curves, for the points indicating the results are shown, and these all fall in so truly that it was hardly necessary to draw a line until the points of low pressure are reached. There is a slight deviation of that part of the curve for hydrogen, stucco No. 1, which represents the pressures below 1 inch; but this has been already explained as being due to the infusion of air through the india-rubber. In order to fully appreciate the force of this agreement, it must be borne in mind that it is not merely the portions of the curves that overlap that agree in direction, but the distance between the curves for hydrogen and air which have been shifted in pairs.

Nothing could prove more forcibly than this, that the different results obtained with different plates are quite independent of the nature of the gas so long as the densities are in the ratio of the fineness of the plates.

So far, therefore, as thermal transpiration is concerned, we have an absolute proof of Law V., Art. 9.

The relative coarseness of the plates.

The shifts in Plate 49 to bring the curves into coincidence being the logarithms of the corresponding pressures, it follows from Law V. that these shifts are the logarithms of the relative coarseness of the plates. Hence for the mean (after some law) diameters of the apertures we have:

Plate.	Coarseness.
Meerschaum No. 3	1
Stucco No. 1	5
Stucco No. 2	5·6

Further comparison of the results with the laws of Art. 9.

32. So far as the manner of variation of the differences of pressures with the density of the gas, this is completely shown by the shapes of the curves in figs. 3, 4, and 5, and is strictly according to Laws II. and III.

The agreement of the log. curves has been shown to confirm Law V. It only remains, therefore, to notice the laws of variation at the greatest and smallest pressures, to see how far these conform to the limits given in Laws III and IV.

According to Law III., when the density of the gas is sufficient, the differences of pressure should be inversely proportional to the density.

Hence, according to this law, the product of the pressure into the difference of pressure should approximate to a constant quantity as the density increases.

In the case of stucco No. 2, we have, adding the two tables of logarithms, subtracting ·684−1=log. ·483, and taking out the numbers

Pressure.	Pressure × difference of pressure÷·483.
30·25	1
30·05	1·004
28·05	·964
27·25	·957
25·85	·940
24·90	·927
23·15	·933

This sufficiently shows that the approximation is very close and according to Law III.

Coming now to the lower pressures, it will at once be seen that in all cases there is a tendency towards constancy. This is best seen in Plate 49, where the curves not only converge towards the left but turn towards the horizontal.

It is clear, however, from these curves, that the limit had not been reached, nor is it possible to say simply from the shape of the curves how far it might be off.

The following comparison, however, will show that the indication is in favour of Law IV., viz.: that the ultimate ratio which the difference of pressure bears to the mean pressure should be as the ratio which the difference of the square roots of the absolute temperature bears to the square root of the mean absolute temperature. According to this, we should have in the case of the meerschaum plate the ratio of the difference of pressure to the mean pressure equal

$$\frac{\sqrt{212+461}-\sqrt{63+461}}{\sqrt{137\cdot5+461}}=\frac{1}{8},$$

whereas, supposing that there was a difference of 20° between the surfaces of the meerschaum and the opposite tin plates

$$\frac{\sqrt{192+461}-\sqrt{83+461}}{\sqrt{137\cdot5+461}}=\frac{1}{11},$$

between which values it is probable that the actual ratio lies.

The highest ratio of the difference of pressure to the mean pressure obtained is 1 to 13, and this may well be considered as an approximation to 1 to 11.

Thus, not only in their general features, but in the approximation towards definite limits, the experimental results show a close agreement with the laws as deduced from the theory.

SECTION III.—EXPERIMENTS RESPECTING THE RATE OF TRANSPIRATION.

33. The experiments to be described in this section, besides being necessary for the verification of the Laws V., VI., and VII., Art. 9, were necessary to complete the verification of Law I. In the last section no direct notice was taken of the rate of thermal transpiration when unprevented by the difference of pressure on the two sides of the plate, and for this reason.

Although the thermal differences of pressure indicate in a general way the manner in which transpiration would have taken place had the pressure been equal, yet in order to examine the results strictly, as regards the various rates of thermal transpiration to which they correspond, it is necessary to know the exact law of transpiration for gases under pressure. The comparative rates of transpiration for different gases and the rates of transpiration of each gas for different pressures are not sufficient. So far, the laws established by GRAHAM are all that can be desired, but these laws say nothing about the variation in the rate of transpiration consequent on a large variation in the density of the gas. Thus, GRAHAM has shown that, through a fine graphite plate, the time of transpiration of a constant volume (measured at the mean pressure) will be exactly proportional to the difference of pressure, and will diminish slightly with the density, but his experiments were not carried to pressures many times less than the pressure of the atmosphere ; whereas, for the purpose of this investigation, it was necessary to compare results at pressures as low as ·01 of an atmosphere. Nor is this the only point in which GRAHAM'S results appeared insufficient for the present comparison. GRAHAM had found that the law of transpiration for a fine graphite plate differed essentially from the law for a stucco plate ; his experiments having been made in both cases at pressure not many times less than the pressure of the atmosphere. Thus, for the stucco plate, the comparative times of transpiration of air and hydrogen were as 2·8 to 1, while for the graphite plate they were as 3·8 to 1. He had also shown that for plates of intermediate coarseness an intermediate ratio would maintain ; but he had given no law that would enable us to predict the result with any particular plate.

In order, therefore, to effect my comparison, it was necessary, by actual experiment, to ascertain the rates of transpiration through my particular plates with the same gases as those used for thermal transpiration, and at similar pressures. It was this consideration which mainly determined the manner of making the experiments.

The apparatus.

34. The thermo-diffusiometer, without the streams of steam and water, after having undergone certain slight modifications, lent itself very well to this part of the investigation.

By means of extra branches from the tube KK, fig. 2, two 8 oz. flasks were connected with the chambers, one on each side of the porous plate, the object of these flasks being simply to enlarge the capacity of the chambers.

The branch to the flask on the right was outside the tap P, so that by closing this tap the flask would be cut off from the instrument, and the action of the pump would be confined to that one flask.

In this condition the mercury pump had a definite capacity—about 6 fluid oz., the capacity of the flasks was definite—about 8 fluid oz. each, and besides these there were the tubes and chambers in the diffusiometer also of definite capacity—about 3 oz. on each side of the plate.

The vacuum gauge was cut off during these experiments, so that the movement of the mercury in the siphon gauge constituted the only source of variation in capacity, and this was small.

This constancy in the capacity of the several parts of the apparatus, if not absolutely essential for these experiments, was very important, as it did away with the necessity of any process of reduction in comparing the results of the experiments at different pressures. This may be seen as follows.

Equal volumes.

35. Starting with the pump full of mercury, and the taps open so that the pressure, whatever it might be, is the same throughout the instrument, both taps being then closed, one stroke of the pump draws a definite proportion of the entire air in the instrument out of the right-hand flask, lowering the pressure in this flask in a definite ratio. Or in other words, one stroke of the pump withdraws from the flask on the right a definite volume of gas as measured at the pressure in the instrument.

This condition would be maintained until the tap P, between the right-hand flask and the instrument, was opened. Then the pressure on the right hand side of the porous plate would fall in a definite ratio. Transpiration would commence, and by the time the pressure on the two sides of the plate had again become equal, a definite volume of air, about half that withdrawn by the pump, must have passed through the porous plate.

The time from the opening of the tap, before complete equalisation is effected, is then seen to be the time of transpiration of a definite volume of gas measured at either the initial or the final pressures in the instrument, under differences of pressure which, although varying, are at corresponding stages proportional to the initial or final pressures in the instrument.

This time, which is called by GRAHAM the time of transpiration of equal volumes, is directly measured in these experiments.

Measurement of the time.

36. The time at which transpiration commenced was the time at which the tap was opened, the tap and the tubes being sufficiently large to allow almost instantaneous

adjustment of the pressures on the right of the porous plate. On first opening the tap P, the mercury in the siphon gauge was displaced, and as equalisation was re-established the mercury re-assumed its level position, the instant of complete transpiration being that at which the mercury became level.

The final adjustment of the mercury, however, was very slow, and it was not found possible, even with the cathetometer, to ascertain definitely the instant of complete equalisation. This threatened to be a difficulty, but it was finally overcome in a very simple manner.

Instead of waiting for complete equalisation, the time was taken at which the equalisation had proceeded, until the residual excess of pressure to the left of the plate bore a certain relation to the initial absolute pressure—·002 was the proportion allowed.

It will be seen that in this way the volume which passed, instead of being the volume for complete equalisation, was some definite proportion of this, and that the differences of pressure under which it passed were proportional to the initial difference of pressure, and hence the time occupied was the time of transpiration of equal volumes according to the previous definition.

The manner of experimenting.

37. The temperature of the room in which the diffusiometer was, having been read, the pump being full of mercury, and the taps D and P open so as to allow of complete equalisation through all the chambers of the instrument, the experiment commenced. The vacuum gauge was read; this gave the initial pressure in the instrument. The position of the mercury on the left side of the differential gauge was then read with the cathetometer.

From this reading was subtracted ·001 of the reading on the vacuum gauge, *i.e.*, the micrometer screw was turned through ten divisions for every inch pressure in the instrument.

The vacuum gauge was then cut off by pinching the india-rubber tubing; the taps P and D closed; one stroke of the pump was taken; a definite volume of air being thus drawn out of the flask, the pump was replaced so as to be full of mercury. Then at a given second, marked by a chronometer, the tap P was opened. A watch was then kept through the cathetometer, until the mercury in the differential gauge descended to line in the cathetometer. As the mercury was still in motion, this instant was well marked by merely raising the eyes to the chronometer.

The small losses of time (personal equations) between reading the chronometer and opening the tap, and reading the cathetometer and the chronometer, were determined as approximately equal to one second, which was accordingly subtracted from the time noticed.

In one set of experiments, that of hydrogen through stucco, the time of equalisation was so small (between 20 and 30 seconds) that a fraction of a second became a matter of some importance, and as the instant at which the eye reached the chronometer did not always correspond with the complete second or half second, there was a liability to this error; but this was to some extent obviated by making successive experiments for such small differences of pressure, that the differences in the reading were much less than a second, and passing over all the observations except those which corresponded with the beat of the chronometer.

With the stucco plates, both for air and hydrogen, three series of readings were taken, and the agreement was found to be very close.

With the meerschaum, the interval of transpiration was so long, about 12 minutes for air and about 3 minutes for hydrogen, that one series of experiments was considered to be sufficient.

It is important to notice here, that while making these experiments I had not the least idea as to how the results would come out when they came to be compared. This comparison was not made for several weeks, as the logarithmic method of comparing them had not occurred to me at the time the experiments were made.

The very remarkable agreement which has been found in the results cannot, therefore, be owing to any bias in my mind, but must be entirely attributed to the accuracy of the means of observation.

Purity of gases.

38. The greatest care was taken to get the gas pure and dry. And as it had been found in the previous experiments that when the pressure in the instrument was low, the gas, particularly the hydrogen, was liable to become contaminated by infusion through the india-rubber, the experiments were not continued to very low pressures and were made as rapidly as possible.

The results of the experiments.

39. Two plates were tried, meerschaum No. 3 and stucco No. 2, which were both in their respective diffusiometers just as they had been used for thermal transpiration. The results are given in the following tables:—

TABLE XIV.—Time of transpiration of equal volumes of air at different pressures through stucco plate No. 2.

Initial pressure.	Time of transpiration in seconds, July, 1878.	Log of pressure.	Log of time.
inches.			
30·10	54·5	1·478	1·7364
29·95	55	1·476	1·7404
26·25	58	1·418	1·7634
26·15	60	1·416	1·7781
22·95	62	1·36	1·7924
20·50	65	1·31	1·8129
19·90	66	1·30	1·8195
17·75	68	1·25	1·8325
15·40	72	1·19	1·8573
13·45	75	1·13	1·8750
11·60	79	1·06	1·8976
10·05	82	1·00	1·9138
8·75	85	·94	1·9294
5·90	90	·77	1·9542
5·35	93	·73	1·9684
5·20	92	·72	1·9638
4·75	95	·68	1·9777
4·50	95	·65	1·9777
3·90	97	·59	1·9867
3·85	97	·58	1·9867
3·40	98	·53	1·9912
2·95	100	·47	2·0000
2·50	101	·40	2·0040
·95	102	·98—1	2·0128

TABLE XV.—Time of transpiration of equal volumes of hydrogen at different pressures through stucco plate No. 2.

Pressures before transpiration.	Time of transpiration in seconds.	Log of pressure.	Log of time.
inches.			
30·10	19	1·48	1·2787
26·30	20	1·42	1·3010
18·6	21	1·27	1·3222
7·35	25	0·87	1·3979
5·50	26	0·74	1·4149
4·15	27	0·61	1·4313
3·75	28	0·57	1·4471
1·35	28·5	0·13	1·4540
1·20	28·5	0·08	1·4540

Table XVI.—Time of transpiration of equal volumes of air at different pressures through meerschaum plate No. 2.

Initial pressure.	Time of transpiration in seconds.	Log of pressure.	Log of time.
inches.			
30·	674	1·477	2·8286
15·10	716	1·179	2·8549
12·75	720	1·105	2·8573
11·25	724	1·051	2·8579
6·40	725	1·806	2·8600
6·00	725	1·792	2·8603

Table XVII.—Time of transpiration of equal volumes of hydrogen at different pressures through meerschaum plate No. 3.

Initial pressure.	Time of transpiration in seconds.	Log of pressure.	Log of time.
inches.			
32·5	187	1·5118	2·27184
13·25	198	1·1222	2·29665
12·40	198	1·0934	2·29666
4·05	200	1·6074	2·30103
3·85	200	1·5854	2·30103

From these tables it appears that the transpiration times at pressures nearly equal to that of the atmosphere are for air and hydrogen, through stucco, as 55 to 19, as 2·9 to 1, while through meerschaum they are as 3·6 to 1.

Graham found the ratio for stucco 2·8 to 1, and for graphite 3·8 to 1.

The small difference between these numbers may be well explained by supposing, as is quite probable, that the stucco used by Graham was rather coarser than plate No. 2, also that the graphite was finer than the meerschaum ; but even allowing the difference, the present results are in very fair accord with Graham's as far as the conditions of pressure corresponded.

When, however, we come to compare the times for air and hydrogen at lower pressures, we see that not only does this ratio differ very greatly from that obtained by Graham for stucco, but that it approaches what he obtained with graphite. Thus at a pressure of 4 inches the ratio of the times are as 96 to 27, as 3·56 to 1, or they are the same as with the meerschaum at the pressure of the atmosphere. For lower pressures we have indications of a still higher ratio. Thus at 1 inch the ratio is 103 to 28·5, or 3·62 to 1.

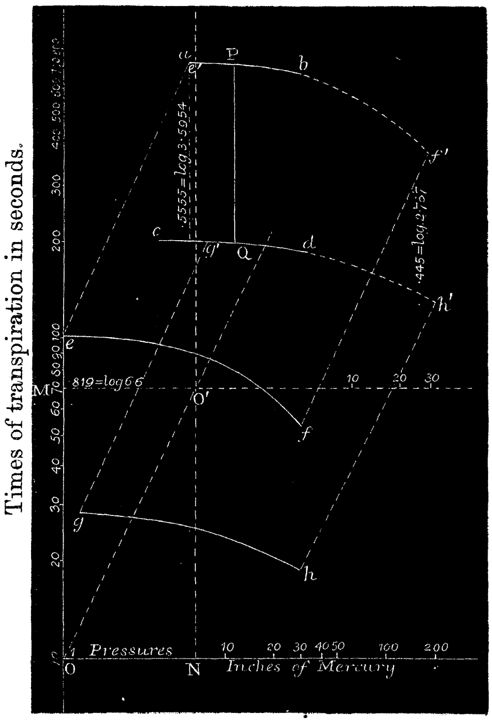
In the same way we see that with the meerschaum as the pressure falls we have an increase in the difference of the times for air and hydrogen.

This variation in the comparative times for air and hydrogen is strictly in accordance with Law VI., Art. 9, as is also the manner of variation, as the pressure falls, of the times for each particular gas. These variations indicate that there are certain pressures for the stucco plate corresponding with certain other pressures for the meerschaum, at which the relation between the times for hydrogen and air are equal, and the variation of these times with the pressure similar.

Logarithmic homologues.

40. To test this, the logarithms of the pressures and times are plotted, and curves drawn, as explained in Art. 28. These are shown in fig. 8.

Fig. 8.



a b and *c d* are the curves for air and hydrogen through meerschaum, *e f* and *g h* are the curves for air and hydrogen through stucco. The figure consisting of the two curves *e f* and *g h* is found to fit on to the figure consisting of *a b* and *c d*, the displacement being from *e f* and *g h* to *e′ f′ g′ h′*. The scale of the figure is too small to allow of the position of the points marking the experiments being shown, but these are shown in the larger figure, diagram 1, Plate 47.

The agreement is there seen to be very close—the very considerable portions of the curves which overlap coming into actual coincidence.

As previously explained with reference to the log. curves for thermal transpiration, the displacement O M in the direction of the abscissæ represents the logarithm of the ratio of corresponding pressures, while the displacement O N in the ordinates represents the log of the ratio of the corresponding times for the two plates. This latter ratio cannot be made use of for the sake of comparison, as it involves the number of openings through the plate as well as their diameters.

The relative coarseness of the plates.

41. The ratio of the corresponding pressures, as given by the difference in the abscissæ, is of the greatest importance. This ratio, according to Law V., Art. 9, corresponds with the ratio for the coarseness of the plates, and as these were the same plates as were used in thermal transpiration, it was to be expected that the results should agree; that is to say, the displacement O′M, fig. 8, should be equal to the displacement O′M, fig. 6. The actual measures give

For thermal transpiration, O′M=·748=log 5·6
,, transpiration under pressure, O′M=·819=log 6·5

By this comparison the two independent and distinct experimental results check one another.

The difference in the results, although too small to cast a doubt upon their agreement, is too large to be attributed to experimental inaccuracy. But it must be remembered that the conditions under which the plates are compared differs in an important particular. In the experiments on thermal transpiration the plates were heated, whereas in the experiments on transpiration under pressure they were at the normal temperatures, and it appears only natural to suppose that such a difference of temperature would somewhat alter the condition of the plate (see Appendix, note 3).

Small densities.

42. It appears very clearly from the curves, that as the pressure of the gas diminishes, the time of transpiration of equal volumes tends to become constant; approximate constancy having been reached in the experiments.

The ultimate ratio of the times of different gases was found by GRAHAM to be as the square roots of the atomic weights of the gases, and the same ratio is obtained for air and hydrogen in these experiments. The square roots of the densities of dry air and hydrogen are 3·8 (3·79) to 1. The ratio of the times for air and hydrogen at the smallest pressures tried is 3·624, and as this is the result for both stucco and meerschaum the approximation is too close to be questioned, particularly when it is remembered that the smallest trace of impurity in the gases might cause the difference.

Large densities.

43. As the density of the gas increases, the times of transpiration diminish, at first slowly, and then more rapidly. According to Law VII., ultimately the time of transpiration becomes inversely proportional to the density: this rate was not reached in the present experiments, the nearest approach being with air through stucco. The shape of the curves, however, shows that the limit has not been reached.

In order, however, to show that the rate of variation of the times of transpiration of equal volumes reaches but does not pass beyond the rate of variation of the inverse density, we have GRAHAM'S experiments on capillary tubes, this being the exact law which was found to hold with all the gases and all the tubes. These tubes may be considered as corresponding with an extremely coarse plate.

GRAHAM'S *results reconciled.*

44. It is thus seen how the apparently different laws obtained by GRAHAM for capillary tubes and plates of different coarseness, which led him to suppose that the passage of the gas through the finer plates more nearly resembled effusion than transpiration, are all reconciled and brought under one general law, involving, besides the nature of the gas, nothing but the ratio which the density of the gas bears to the fineness of the plate.

The verification of Law I.

45. The deduction of the comparative rates of thermal transpiration which would have ensued had the tap D in the thermo-diffusiometer have been open, is now only a matter of calculation. We have only to calculate by Law VI. the comparative rates of transpiration that would have resulted from the thermal differences of pressure. Hence it will be seen that Law I. follows from Laws II. and VI., Art. 9, and as these have both been verified, Law I. has also been verified.

SECTION IV.—EXPERIMENTS WITH VERY SMALL VANES.

First experiments.

46. Before commencing the experiments on thermal transpiration described in Section II., I made an attempt to ascertain how far were borne out the theoretical conclusions that the necessity for extremely small pressures in the radiometer was owing to the comparatively large size of the vanes, and that with smaller vanes similar results would be obtained at proportionally higher pressures.

The pressures at which the impulsive force in the radiometer first becomes sensible is so extremely small that this pressure may be increased several hundred fold without becoming what may be called sensible—measureable by a mercurial gauge. So that on the assumption that the pressure at which the effect would be apparent, increases proportionally as the size of the vanes diminishes, it was clear that in order to obtain the repulsive effect at the pressure of the atmosphere the size of the vanes must be reduced several thousand times.

The only means of obtaining such small vanes was to suspend a fibre of silk or a spider line. A single fibre of silk has a diameter of $\frac{1}{2000}$th of an inch (about), which is less than $\frac{1}{1000}$th the breadth of the vanes of the light mill on which my previous experiments had been made. But in order that the pressures at which the results would be sensible might be inversely proportional to the size of the vanes, the vanes should preserve the same shape; whereas the vanes in the light mill were square, while the fibre of silk was only narrow in one direction, which would be considerably to the disadvantage of the fibre of silk. More than this: it appeared probable that the thinness and transparency of the fibre, together with the cooling action of the air, would only allow an extremely small difference of temperature to be maintained on its opposite faces by radiant heat falling on one side; whereas air currents in the tubes, which would tend to carry the fibre with them, would be caused by the greater temperature of the glass on that side of the tube on which was the hot body, and these, which would be quite independent of the size of the fibre or vane, would exercise, proportionally, as great an effect on the fibre as on the larger vanes.

For the foregoing reasons a result was hardly probable, but as a preliminary step I suspended a fibre in a test tube ·7 inch in diameter and 5 inches long; I then brought a gas flame near to the tube to see if it would cause any motion in the fibre, the pressure of the air within the tube being that of the atmosphere.

The result was that the hair moved very slightly and somewhat uncertainly towards the flame.

As I had more than suspected that such would be the result at the pressure of the atmosphere, and as I had no means at hand for exhausting the tube, I postponed further experiments in this direction in order to take up the more promising investigation with the porous plates. When, however, I had concluded this, and succeeded almost beyond my expectation, I returned to the experiments on the fibre with the intention of exhausting the tube and using hydrogen as well as air.

Subsequent experiments.

47. These experiments were commenced on July 24, 1878.

A single fibre of unspun silk, having a thickness of ·0005 of an inch, was suspended in a test tube 1 inch in diameter and 7 inches long. The tube was closed with an india-rubber cork, through which passed a small glass tube to allow of exhaustion; this tube was connected with the vacuum gauge and the mercury pump, also with drying tubes for admitting dry air or hydrogen. A microscope with micrometer eyepiece reading $\frac{1}{10000}$th of an inch (the same as had formed the cathetometer in the previous experiments) was arranged for the observation of the motion of the fibre.

The apparatus as arranged is shown in fig. 9, Plate 47.

The tube having been dried was filled with dry air at the pressure of the atmosphere. A hot body was then brought near it.

In order to secure uniformity in the hot body, a test tube filled with boiling water was placed on a stand, which stand remained in the same position throughout the experiment, the water in the test tube being boiled the instant before the tube was placed on the stand.

The motion of the fibre was then watched through the microscope and measured.

Having ascertained the motion, the heater was removed and the fibre allowed to return to its normal position, which it always did with more or less exactness.

The tube was then exhausted to a limited extent and the operations repeated.

48. In this way were obtained a series of observations both for air and hydrogen at various pressures. These are shown in the following tables.

TABLE XVIII.—Impulsion of fibre of silk in air, August 1, 1878.

Pressure by vacuum gauge.	Motion of the fibre.
inches.	inches.
30	—·0930
16	—·0300
8	+·00
4	+·0150
2	+·0210
1	+·0230
·5	+·0390
·2	+·0700
·1	+·0830
·05	+·0930
·025	+·1005

TABLE XIX.—Impulsion of fibre of silk in hydrogen, August 1, 1878.

Pressure by vacuum gauge.	Motion of the fibre.
inches.	inches.
30	+·0040
16	+·0070
8	+·0100
5	+·0160
2	+·0310
1	+·0490
·40	+·0710
·2	+·0880
·1	+·1070

Table XVIII. shows that with air the result was negative until a pressure of less than 8 inches was obtained, it then became positive, and it was measurable at a pressure of 4 inches, and then steadily increased as the pressure fell, until for very small pressures the fibre moved through about 1,000 divisions on the micrometer.

With hydrogen, Table XIX. shows that the results were positive from the pressure of the atmosphere and for small pressures were somewhat larger than with air.

Although only one series of such observations is recorded in the table, the experiments were repeated several times with each gas. Also a flame was used instead of a heater, and the results were consistent throughout.

Elevation of the heater.

49. The effect of having the heater at different elevations was carefully studied, for it was obvious that this would affect the air currents in the tube. It was found, however, that the elevation of the heater did not produce any effect on the direction in which the fibre moved at pressures of less than 6 or 8 inches of mercury for air, and less than 20 inches for hydrogen. For pressures greater than these, considerable alterations in the elevation of the heater did produce very slight modifications in the motion of the fibre.

Bending of the fibre.

50. The possibility of the results being due to a tendency of the fibre to bend with the warmth was also considered. Observations were taken at different points up the fibre and on different sides ; and the results were such as to lead to the conclusion that the bending of the fibre did not produce any material effect.

Spider line.

51. A spider line was also used : it was not found possible to suspend this freely in the tube. It was attached top and bottom to a wire frame, but it was quite loose between the points of attachment, so that it could swing to either side.

Considerable difficulty was found in observing the spider line, as it was lost sight of the instant it was the least out of focus ; but the general result of the observation was, that at higher pressures both for air and hydrogen the motion was negative or to the heater ; but at pressures of less than about 8 inches it was decidedly positive, the fibre being driven away from the heater as far as its frame would allow.

From the fact that the fibre of silk had shown positive motion so nearly up to the pressure of the atmosphere it might have been anticipated that the spider line, on account of its much greater thinness, would have shown positive motion even at pressures considerably above that of the atmosphere. But the reasoning of Art. 46 respecting the differences of temperature to be maintained and the effect of the air currents, obviously applies with greater force to the spider line than to the fibre of silk, and at once accounts for the observed fact that the positive motion with the spider line was not obtained until the pressures were somewhat lower than those necessary for the fibre of silk.

52. Both with the fibre of silk and the spider line the phenomena of impulsion (the excess of pressure against warm surfaces) were apparent and consistent at densities many hundred times greater than the highest densities at which like results are obtained with vanes several hundred times broader than the fibre of silk; this verifies the theoretical conclusion on which this part of the investigation was based. The results in this case are not so definite as is the agreement of the logarithmic homologues in the instances of transpiration; but the one fact supports the other, and we may consider the law of impulsion—Law VIII., Art. 9—to have been sufficiently proved.

This concludes the experimental investigation.

PART II.—(THEORETICAL).

SECTION V.—INTRODUCTION TO THE THEORY.

53. In suggesting in a former paper that the results discovered by Mr. CROOKES were due to the communication of heat from the surface of the solid bodies to the gas surrounding them, I pointed out as the fundamental fact on which I based my explanation, that when heat is communicated from a solid surface to a gas, the mean velocity of the molecules which rebound from the surface must be greater as they rebound than as they approach, and hence the momentum which these particular molecules communicate to the surface must be greater than it would be if the surface were at the same temperature as the gas.

So far the reasoning is incontrovertible. But in order to explain the experimental results, it was necessary to assume that the number of cold molecules which approached the hot surface would be the same as if the surface were at the same temperature as the gas, or at any rate if reduced the number would not be sufficiently reduced to counteract the effect of increased velocity of rebound.

Although at that time I could not see any definite proof of this, nor any way of definitely examining the question, yet I had a strong impression that the assumption was legitimate; and although I hoped at some future time to be able to complete the theoretical explanation, I was content for the time to rest the evidence of the truth of the assumptions involved on the adequacy of the reasoning to explain the experimental results obtained.

As other suggestions respecting the cause of the phenomena, widely different in character from mine, had found supporters, and a good deal of scepticism was expressed as to the fitness of the cause which I had suggested, my attention was occupied in deducing the actions which must result from such a force, and comparing them with experimental results. Having, however, at length satisfied myself, and seeing that a conviction was spreading that what I suggested contained the germ of the explanation, I set to work in earnest to complete the explanation, and ascertain by an extension of the dynamical theory of gases what effect the hot molecules receding from the surface should produce on the number and temperature of those approaching.

My first attempts to accomplish this were altogether unsuccessful. When contemplating the phenomena it seemed to me that I could perceive a glimmering of the method of reasoning for which I was in search, but as soon as ever I attempted to give definite expression to it this glimmering vanished.

The reason for this I now perceive clearly. When contemplating the phenomena, I had a surface of limited extent before me, and I considered the effect on such a surface without recognising the fundamental importance of the limit to size.

On the other hand, when I came to definite reasoning, for the sake of what appeared to be a simplification of the conditions of the problem, I assumed the surface to be without limit, thus introducing a fundamental alteration into the conditions of the problem without perceiving it.

The importance of this limit only became apparent to me when I found, by simple dynamical reasoning, that with surfaces of unlimited extent such results as those actually obtained would be impossible. This appeared as follows :—

No force on unlimited surface.

54. If we had two plane plates of unlimited extent, H and C, the surface of H opposite to C being hotter than the surface of C which was opposite to H, the outside surfaces of both plates being at the same temperature, then in order to produce results similar to those obtained with limited plates, the gas between the two plates must maintain a greater steady pressure on the plate H, than that which it exerts on the colder plate C. Whereas it is at once obvious that such a condition is contrary to the laws of motion, which require that the gas between the two surfaces should exert an equal and opposite pressure on both surfaces.

Having once perceived the force of this reasoning, it became clear to me that if, as I had supposed, the results obtained in the experiments were due to gaseous pressure, then they must depend on the limited extent of the surfaces.

This gave me the clue, in following which I have not only had the satisfaction of finding the explanation complete as regards the phenomena from which it originated, but I have also found that the theory indicated the phenomena of thermal transpiration, and explains much that hitherto has been considered anomalous respecting the laws of transpiration of gases through small channels—suggesting the experiments by which might be established the relation between these actions.

The manner in which the force arises in the case of a limited surface was at first rendered much clearer to me by considering an illustration, which I introduce here, although it forms no part of the proof which will follow.

55. Instead of H and C being plates with gas between them, let them be earthen batteries of *unlimited length,* and suppose that guns are distributed at uniform intervals along those batteries; suppose, also, that all the shot fired from H bury themselves in the earth of C, and *vice versâ.*

Then, in the first place, it is obvious that since on firing a shot the momentum imparted to the gun is equal and opposite to the momentum given to the shot, every shot fired from H will exercise the same force to move the battery H away from C as the shot will exercise to move C away from H; and in the same way the recoil of the guns on C will exercise the same tendency to move C away from H as the shot will exercise to move H away from C. And this will be the case whether the guns are supposed to be pointed straight across the interval between the batteries, or, *as I shall suppose, are pointed with various degrees of obliquity.*

Since, then, the result of every shot, whether fired from H or C, causes equal and opposite forces on the two batteries, the result of all the firing, no matter how much harder one battery may bombard than the other, must be to cause an equal force on each battery, the batteries being of unlimited length.

This case will be seen to be strictly analogous to the effect of the gas between two plates of unlimited extent to cause equal pressures on the plates, no matter what may be the differences in the temperature of the plates.

If now we consider the batteries of limited extent, then, owing to the obliquity of the guns, some of the shot from H may pass beyond the ends of C, and *vice versâ;* and in this case the force of recoil on the battery which fires will no longer be balanced by the stopping of the shot on the other battery. So that supposing the directions of firing to be similar, that battery which fires the hardest will be subject to the greatest tendency to move back.

The battery which fires the hardest corresponds with the hottest plate; and hence we perceive by analogy that, if of limited extent, the hottest plate will experience the greatest pressure from the gas between the plates.

56. The analogy between the batteries and the plates is rendered more strict if we suppose the batteries H and C to be two limited batteries, each placed in front of a battery of unlimited extent, and that these unlimited batteries are pounding away in an exactly similar manner.

The effect of the shot from these unlimited batteries on H and C will be analogous to the effect of the gas outside and beyond the plates. And it is at once seen that these unlimited batteries will produce similar effects on H and C respectively, and that the effect of the firing between H and C will be uninfluenced by the batteries behind, and therefore, as before, that battery will be subject to the greatest tendency to move back which fires the hardest.

To make the analogy between the two cases complete, suppose that H and C, in addition to pounding away at each other, are exactly returning the fire of the batteries from behind, and that the mean rate at which H fires at C and C at H are exactly the same as the rate at which the other firing goes on, but that the velocity of the shot from H is just as much greater than the mean velocity, as the velocity of the shot from C is below the mean. Then it is at once seen that the total tendency on H is to move back, while the total tendency on C is to move forward.

It obviously follows from the foregoing that the inequality in the forces on H and C could only occur at a certain distance from their ends, which distance would depend on the distance between the batteries; and hence that the ratio which this inequality (due to any particular rate of firing) would bear to the whole reaction on either battery would increase as the length of the batteries diminished; or in other words, the inequality of force would be proportional to the distance between the batteries, and would be constant whatever might be the length of the batteries beyond a certain point.

At first sight it may appear that the distance between the batteries H and C should be analogous to the distance between the hot and cold plates; but it is necessary to remember that it is only in case of the gas being extremely rare, as compared with the distance between the plates, that the molecules can be supposed to go straight from the one plate to the other. In ordinary cases the molecules encounter other molecules, and the effect of such encounters is to reduce the motion to a mean. Hence it appears that the distance between the batteries as affecting the equality in the reactions is somewhat analogous to the distance which a molecule may be supposed to travel without losing its characteristic motion. And hence it would appear that in the case of gas the inequalities of force on the two plates would be proportional to the inverse density of the gas and the extent of the boundaries of plates.

57. The shot from H which miss C, and those from C which miss H, must be stopped by the outside batteries. Therefore the inequalities in the forces on H and C will be balanced by inequalities in the forces on the batteries behind, and the sum of the forces on H and the battery behind will be equal to the sum of the forces on C and the battery behind.

And this is strictly analogous to the result of SHUSTER'S experiment, viz.: that the effect upon the vanes of the light mill is exactly balanced by the effect on the containing vessel.

58. The batteries also serve to illustrate the action of thermal transpiration. In the case already considered (Art. 57) the inequality between the shot from H which miss C and those from C which miss H is transferred to the outside batteries, or in the case of the gas, to the containing vessel. The better to illustrate the present point, suppose that the outside batteries are ranged across the ends of the open space between H and C. This will make no difference to the result. The inequality of the action of the shot which miss H and C must now cause a force parallel to the end batteries, tending to cause these batteries to move end-wise in the direction of C.

Suppose that the two batteries H and C were free to move together in the direction from C to H (suppose them on a truck). The inequality in the force would set them in motion in this direction, which motion would increase until the actual velocity of the shot from C equalled the actual velocity of the shot from H; then all inequalities in the reactions would cease, and there would be no reactions on the limiting batteries.

In this case the limiting batteries are obviously analogous to the sides of a tube, and the interval between the planes H and C corresponds with a layer of gas at equal pressures, but across which the heat is being conducted by the greater velocity of the molecules which move from H to C; and the conclusion is that such a layer of gas when maintained at rest exerts a tangential force on the sides of the tube tending to move the tube in the direction of the flow of heat, whereas if the gas were free to move it would flow towards the hottest end; and this is the phenomenon of thermal transpiration.

59. The foregoing illustration, with the exception that the action is confined to a plane instead of being distributed through a space, is more than analogous: it is strictly parallel to the case of gas as long as the gas is so rare that the molecules proceed straight across the intervals between the plates or sides of a tube. When this is the case, therefore, the example of the batteries explains the phenomena of thermal transpiration as well as the phenomena of the radiometer. But when the gas is so dense that in crossing the interval between the surfaces the molecules undergo several encounters, the parallelism no longer holds. Even then, however, the analogy holds, for the gas at any point may be considered as consisting of two sets of molecules which are moving across a plane from opposite sides. And by examining the difference in the velocity of these two sets of molecules a general explanation of many of the phenomena may be obtained without recourse being had to a strict analytical investigation. The analogy has, however, been pursued far enough to serve the purpose of an introduction.

Before proceeding to the mathematical investigation, which is novel and somewhat intricate, I have thought it advisable to further introduce it by a short description of the method used and the assumptions involved.

Prefatory description of the mathematical method.

60. The characteristic as well as the novelty of this investigation consists in the method by which not only the mean of the motions of the molecules at the point under consideration is taken into account, but also the manner in which this mean motion may vary from point to point in any direction across the point under consideration. It appears that such a variation gives rise to certain stresses in the gas (tangential and normal), and it is of these stresses that the phenomena of transpiration and impulsion afford evidence.

Instead of considering only the condition of the molecules comprised within an elementary unit of volume of the gas, what is chiefly considered in this investigation is the condition of the molecules which cross a plane supposed to be drawn through the point, which plane may or may not be in motion along its normal.

The molecules which cross this plane are considered as consisting of two groups, one crossing from the positive to the negative side of the plane, and the other crossing from the negative to the positive side. Considered in opposite directions, the mean characteristics (the number, mass, velocity, momentum, energy, &c.) of these two groups are not necessarily equal: they may differ in consequence of the motion of the gas, the motion of the plane through the gas, or a varying condition of the gas. And the determination of the effects of these causes on the mass, momentum, and energy that may be carried across by either group is the more general result of the investigation.

61. As a preliminary step, it is shown that whatever may be the nature of the encounters between the molecules within a small element, the encounters can produce no change on the mean component velocities of the molecules which in a definite time pass through the element; and hence, whatever may be the state towards which the encounters tend to reduce the gas, this state must be such that the mean component velocities of the molecules which pass through the element in a unit of time remain unaltered. These mean component velocities, it is to be noticed, are not the mean component velocities of the molecules within an element at any instant.

Certain assumptions are then made. These do not involve any law of action between the molecules. They are equivalent to assuming that the tendency of the encounters within an element is to reduce the gas to a uniform state.

From these assumptions two theorems (I. and II.) are deduced. From theorem I. it follows that the rate of approximation to a uniform gas is inversely proportional to a certain distance s, which distance is inversely proportional to the density and is some unknown function of the mean velocity of the molecules. From theorem II. it follows that the molecules which enter a small element from any particular direction arrive as if from the uniform gas to which the actual gas tends at a point distant s in the direction from which the molecules come.

When the gas is continuous about the element for distances large compared with s, then s is independent of the direction from which the molecules come; but near a solid surface s is a function of this direction and of the position of the element with respect to the solid surface.

These theorems are fundamental to all the reasoning which follows; and the distance s enters as a quantity of primary importance into all the results obtained.

It is proposed to call this distance the *mean range* of the characteristics of the molecules. Thus we have the mean range of the mass, the mean range of momentum, and the mean range of energy. By qualifying the term "mean range" by the name of the quantity carried, instead of considering it as a general characteristic of the condition of the gas, two things are avoided—

(1) It is not implied that the mean range is the same for all the quantities which may be considered;

(2) There is no fear of confusing the mean range with the mean path of a molecule.

The mean range, whatever may be the nature of the quantity considered, is obviously a function of the mean path of the molecules, and is a small quantity of the same order as the mean path, but it also depends on the nature of the impacts between the molecules.

The symbol s is used to express the mean range of any particular quantity Q.

62. Assuming that the mean value of Q for the molecules in an elementary unit of volume at a point is a function of the position of the point, the aggregate value of Q carried across the plane at a point is obtained in a series of ascending powers of s. And by neglecting the terms which involve the higher powers of s, which terms also involve differentials of Q of orders and degrees higher than the first, equations are obtained between s and the aggregate value of Q carried across the plane.

63. The dynamical conditions of steady momentum, steady density, and steady pressure are next considered. General equations are obtained for these conditions, which general equations involve s, the motion of the plane and other quantities depending on the condition of the gas.

The condition that there may be no tangential stress in the gas is also considered.

It is found that when there is no tangential stress on a solid surface wherever it may be in the gas, the mean component velocities of all the molecules which pass through the element in a definite time must be zero at all points in the gas.

64. The equations of motion are then applied to the particular cases which it is the object of this investigation to explain. Two cases are considered. The first, that of a gas in which the temperature and pressure only vary along one particular direction, so that the isothermal surfaces and surfaces of equal pressure are parallel planes; this is the case of transpiration. The second case is that in which the isothermal surfaces and the surfaces of equal pressure are curved surfaces (whether of single or double curvature); this is the case of impulsion and the radiometer.

As regards the first case, the condition of steady pressure proves to be of no importance; but from the conditions of steady momentum and steady density an equation is obtained between the velocity of the gas, the rate at which the temperature varies, and the rate at which the pressure varies; the coefficients being functions of the absolute temperature of the gas, the diameters of the apertures, and the ratio of the diameters of the apertures to the mean range. These coefficients are determined in the limiting conditions of the gas, when the density is small and large, and as they vary continuously with the condition of the gas, the limiting values afford indications of what must be the intermediate values.

From this equation, which is the general equation of transpiration, the experimental results, both as regards thermal transpiration and transpiration under pressure, are deduced.

In dealing with the second case, that in which the isothermal surfaces are curved, the three conditions—steady momentum, density, and pressure—are all of them important. These conditions reduced to an equation between the motion of the gas, the variation in the absolute temperature, and the variation in pressure, in which, as in the equation of transpiration, the coefficients are functions of the absolute temperature, the diameters of the apertures, and the ratios of the diameters of the apertures to the mean range.

The reduction of the conditions of equilibrium to this equation, however, involves the assumption that the gas should not be extremely rarefied. In order to take this case into account a particular example is examined, and the equation so obtained, together with the equation obtained from the conditions of steady motion, is shown to lead to the results of impulsion and the phenomena of the radiometer.

SECTION VI.—NOTATION AND PRELIMINARY EXPRESSIONS.

65. In arranging the notation I have endeavoured as far as possible to make it similar to the notation already adapted to the kinetic theory of gases by previous writers. With this object I have adopted almost entirely, both as regards symbols and expressions, the notation used by Professor MAXWELL in his paper "On the Dynamical Theory of Gases."* But his notation, copious as it is, has fallen far short of my requirements. I have had to take under consideration certain quantities which have not hitherto been recognised; and what has particularly taxed my resources in symbolising, is that I have had, according to my method, to devise symbols to express each of twenty-four partial or component quantities which spring from any one of certain quantities, which have hitherto been dealt with as simple quantities.

Explanation of the symbols.

66. u, v, w, are used to represent the component velocities of a molecule with reference to the fixed axes x, y, z.

ξ, η, ζ are used to represent the component velocities of a molecule with reference to axes parallel to x, y, z, but which move with the halves of the mean component velocities of the molecules which pass through an element in a definite time.

U, V, W are used to represent the component velocities of the moving axes.

Throughout this investigation bars over the symbols indicate the mean taken over some group of molecules; when no further indication as to the particular group is given, it is to be understood that the mean is taken from the entire group in a unit of

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* Phil. Trans. 1867.

volume at the same instant. Thus $\overline{\xi^2}$, $\overline{\eta^2}$, $\overline{\zeta^2}$ indicate the mean squares of ξ, η, ζ respectively for all the molecules in a unit of volume of uniform gas which is in the same mean condition as the gas at the point considered.

Q is used to represent any quantity belonging to a molecule, such as its mass, momentum, energy, &c.

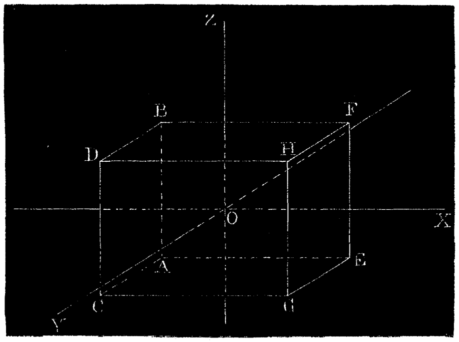
$\Sigma(Q)$ is used to represent the aggregate value of Q for a group of molecules as existing in a unit of volume; and when no further indication is given it will be understood that the aggregate is that of the entire group.

$\sigma(Q)$ indicates the aggregate value of Q carried across a unit of plane area in a unit of time by a group of molecules, which in the absence of further indication will be understood to be the entire group which crosses the plane.

$\sigma_x(Q)$, with the suffix, is used to express the direction of the plane as well as the aggregate value carried across it.

$\overset{u+}{\sigma_x}(Q)$, with the superimposed symbol, expresses the group over which the summation extends; $u+$ indicates that the summation is taken over all those molecules which are moving in the positive direction as regards the axis of x. By varying the superimposed symbol, the general symbol may be made to express the value of Q carried by a group of molecules having any particular motion across the plane indicated by the suffix.

Fig. 10.



67. As indicated by the signs of the component velocities, the molecules in a unit of volume or the molecules which cross a surface at a point in a unit of time will be divided into eight groups.

These groups may be indicated by the eight corners of a cube, having its edges parallel to the axes, circumscribed about the point considered. Thus in fig. 10 the group which have $u+$, $v+$, $w+$, will approach O from the region indicated by the corner A, and similarly there will be a corner for each group. The particular groups, therefore, may be distinguished by the letters at the corners of the cube, fig. 10. And instead of

$$\overset{\substack{w+\\v+\\u+}}{\Sigma}(Q),\ \overset{\substack{w-\\v+\\u+}}{\Sigma}(Q),\ \overset{\substack{w+\\v-\\w+}}{\Sigma}(Q),\ \overset{\substack{w-\\v-\\w+}}{\Sigma}(Q),\ \overset{\substack{w+\\v+\\w-}}{\Sigma}(Q),\ \overset{\substack{w-\\v+\\u-}}{\Sigma}(Q),\ \overset{\substack{w+\\v-\\w-}}{\Sigma}(Q),\ \overset{\substack{w-\\v-\\u-}}{\Sigma}(Q)$$

we have respectively,

$$\overset{a}{\Sigma}(Q),\ \overset{b}{\Sigma}(Q),\ \overset{c}{\Sigma}(Q),\ \overset{d}{\Sigma}(Q),\ \overset{e}{\Sigma}(Q),\ \overset{f}{\Sigma}(Q),\ \overset{g}{\Sigma}(Q),\ \overset{h}{\Sigma}(Q).$$

And in order still further to simplify the notation, instead of

$$\overset{a}{\sigma}(Q),\ \overset{b}{\sigma}(Q),\ \overset{c}{\sigma}(Q),\ \overset{d}{\sigma}(Q),\ \overset{e}{\sigma}(Q),\ \overset{f}{\sigma}(Q),\ \overset{g}{\sigma}(Q),\ \overset{h}{\sigma}(Q)$$

we may write respectively the simple letters

A, B, C, D, E, F, G, H.

68. The method of considering the value of Q carried across a plane by groups of molecules distinguished by the directions in which they are moving, constitutes the essential means by which the results of this investigation are arrived at. And as it does not appear that this method has been resorted to by any previous writer, it appears necessary for me to describe at some length the preliminary steps.

The rate at which Q is carried across a plane.

69. Since the aggregate value of Q carried across any plane by the entire group of molecules must be equal to the sum of the values of Q carried across by all the various groups into which the gas may be divided, we have

$$\sigma(Q)=A+B+C+D+E+F+G+H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\overset{u+}{\sigma}(Q)=A+B+C+D \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\overset{u-}{\sigma}(Q)=E+F+G+H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\overset{v+}{\sigma}(Q)=A+B+E+F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\overset{v-}{\sigma}(Q)=C+D+H+G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\overset{w+}{\sigma}(Q)=A+C+E+G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\overset{w-}{\sigma}(Q)=B+D+F+H \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Gas in uniform condition.

70. When the gas is uniform, whether at rest or in motion, the value of $\sigma(\mathrm{Q})$ has already been determined by Professor MAXWELL, but it is necessary to transform the expressions to the notation of this paper.

We have by a well-known formula*

$$\left.\begin{aligned} \mathrm{A}_x &= \Sigma(\overset{a}{u}\mathrm{Q}), \\ \mathrm{A}_y &= \Sigma(\overset{a}{v}\mathrm{Q}), \\ \mathrm{A}_z &= \Sigma(\overset{a}{w}\mathrm{Q}), \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (8)$$

in which the suffixes x, y, z, indicate that it is the planes yz, zx, xy, that Q is being carried across, and the superimposed symbols a a a indicate the group of molecules over which the summation extends.

We have also

$$\left.\begin{aligned} \mathrm{B}_x &= \Sigma(\overset{b}{u}\mathrm{Q}), \\ \mathrm{B}_y &= \Sigma(\overset{b}{v}\mathrm{Q}), \\ \mathrm{B}_z &= \Sigma(\overset{b}{w}\mathrm{Q}), \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (9)$$

and similar expressions for the values of Q carried across each of the other planes by all the other groups.

In the equation (8) and similar equations we may obviously substitute for u, v, w their values

$$\left.\begin{aligned} u &= \xi + \mathrm{U} \\ v &= \eta + \mathrm{V} \\ w &= \zeta + \mathrm{W} \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (10)$$

And since the gas is here supposed to be uniform, we shall have

$$\left.\begin{aligned} \mathrm{U} &= \bar{u} \\ \mathrm{V} &= \bar{v} \\ \mathrm{W} &= \bar{w} \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (11)$$

ξ, η, ζ being identically the same as if the gas were at rest.

71. For the purpose of this investigation it is necessary to express such quantities as $\overset{u+}{\sigma_x}(\mathrm{Q})$, $\overset{u-}{\sigma_x}(\mathrm{Q})$ in terms of the groups distinguished by the signs of ξ, η, ζ, instead of

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* "On the Dynamical Theory of Gases," MAXWELL; Phil. Trans. 1867, p. 69.

u, v, w; and owing to the fact that in all the cases to be considered U^2, V^2, W^2 are of the second order of small quantities compared with $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$ this may be done. For we may put

$$\overset{u+}{\sigma_x}(Q)=\Sigma\{(\xi\overset{u+}{+}U)Q\}=\Sigma\{(\xi\overset{\xi+}{+}U)Q\}+\Sigma\{(\xi\overset{u+\text{ and }<U}{+}U)Q\} \quad . \quad . \quad . \quad . \quad (12)$$

$$\overset{u-}{\sigma_x}(Q)=\Sigma\{(\xi\overset{u-}{+}U)Q\}=\Sigma\{(\xi\overset{\xi-}{+}U)Q\}-\Sigma\{(\xi\overset{u+\text{ and }<U}{+}U)Q\} \quad . \quad . \quad . \quad . \quad (13)$$

and when U is small compared with $\sqrt{\overline{u^2}}$, the last term on the right in each of these equations will be small to the second order as compared with the first term. For the number of molecules over which the summation in these terms extends is to the whole number of molecules in a unit of volume in something less than the ratio of U to $\sqrt{\overline{u^2}}$. Hence, as will subsequently appear, in neglecting these last terms we shall be neglecting nothing within the limits of our approximation. We have therefore

$$\left.\begin{aligned}\overset{u+}{\sigma_x}(Q)&=\Sigma\{(\xi\overset{\xi+}{+}U)Q\}\\ \overset{u-}{\sigma_x}(Q)&=\Sigma\{(\xi\overset{\xi-}{+}U)Q\}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and similarly for all other groups. Thus it appears that the letters a, b, c, &c., may be used indifferently to indicate the groups as distinguished by the signs of u, v, w or of ξ, η, ζ.

Distribution of velocities amongst the molecules.

72. Although not actually essential to this investigation, as it will tend greatly to simplify the results obtained, I shall adopt the conclusion arrived at by Professor MAXWELL* with respect to the distribution of velocities amongst the molecules of a uniform gas, viz.:—

$$dN=\frac{N}{\alpha^3\pi^{\frac{3}{2}}}e^{-\frac{\xi^2+\eta^2+\zeta^2}{\alpha^2}}d\xi d\eta d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where N is the whole number of molecules in a unit of volume, and dN the number whose component velocities lie between ξ and $\xi+d\xi$, η and $\eta+d\eta$, and ζ and $\zeta+d\zeta$.

From equation (15) we have for a uniform gas

$$\overset{a}{\xi}=-\overset{e}{\xi}=\frac{\alpha}{\sqrt{\pi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

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* Phil. Trans., 1867, p. 65.

$$\overset{a}{\overline{\xi^2}}=\overset{e}{\overline{\xi^2}}=\frac{a^2}{2}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

$$\overset{a}{\overline{\xi^3}}+\overset{a}{\overline{\xi\eta^2}}+\overset{a}{\overline{\xi\zeta^2}}=\frac{3}{\sqrt{\pi}}\alpha^3\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

$$\overset{a}{\overline{\xi\eta}}+\overset{b}{\overline{\xi\eta}}=\overset{a}{\overline{\eta\zeta}}+\overset{e}{\overline{\eta\zeta}}=\overset{a}{\overline{\zeta\xi}}+\overset{e}{\overline{\zeta\xi}}=\frac{\alpha^2}{\pi}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

Also if τ is the absolute temperature of the gas, p the intensity of pressure, M the mass of a molecule, and ρ the density of the gas, we have for uniform gas

$$\frac{\tau}{\mathrm{M}}=\kappa^2\alpha^2\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

$$p=\rho\frac{\alpha^2}{2}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

$$\frac{p}{\rho}=\frac{2}{\kappa^2}\frac{\tau}{\mathrm{M}}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

in which κ^2 varies with the nature of the gas, and is otherwise constant.

73. The adoption of equations (15) to (22) restricts the application of the results that may be arrived at to gases of uniform molecular texture such as air and hydrogen. For these equations do not apply to a varying mixture of gases. In order to render them applicable to such a mixture it would be necessary to consider throughout the investigation the presence of at least two systems of molecules. This would add greatly to the complication, whereas none of the experimental results which it is my immediate object to explain involve a varying mixture.

It will be seen, however, that at least one important result which has not hitherto been explained could be fully explained in this way. This is the transpiration of a varying mixture of two gases through a porous plate. The possibility of such an explanation will be seen from the results obtained for a simple gas.

74. Table XX. contains all the value of $\sigma(\mathrm{Q})$ carried across the axial planes by the several groups of molecules in a uniform gas for all the quantities Q which are important in this investigation.

TABLE XX.—Showing the values of Q carried across the axial planes by the several groups of molecules.

	Q=M.	Q=Mu.	Q=Mv.	Q=Mw.	Q=M$(u^2+v^2+w^2)$.
A_x	$\frac{\rho}{8}\left(U+\frac{\alpha}{\sqrt{\pi}}\right)$	$\frac{\rho}{8}\left(\frac{\alpha^2}{2}+2\frac{\alpha U}{\sqrt{\pi}}\right)$	$\frac{\rho}{8}\left(+\frac{\alpha V}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}+\frac{\alpha U}{\sqrt{\pi}}\right)$	$\frac{\rho}{8}\left(+\frac{\alpha W}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}+\frac{\alpha U}{\sqrt{\pi}}\right)$	$\frac{\rho}{8}\left(\frac{5}{2}\alpha^2 U+\frac{3\alpha^3}{\sqrt{\pi}}+\frac{2\alpha^2}{\pi}(V+W)\right)$
B_x	+	+	+ + +	+ − −	+ + −
C_x	+	+	+ − −	+ + +	+ − +
D_x	+	+	+ − −	+ − −	+ − −
E_x	−	−	− − +	− − +	− − −
F_x	−	−	− − +	− + −	− − +
G_x	−	−	− + −	− − +	− + −
H_x	−	−	− + −	− + −	− + +
A_y	$\frac{\rho}{8}\left(V+\frac{\alpha}{\sqrt{\pi}}\right)$	$\frac{\rho}{8}\left(+\frac{\alpha U}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}+\frac{\alpha V}{\sqrt{\pi}}\right)$	$\frac{\rho}{8}\left(\frac{\alpha^2}{2}+\frac{2\alpha V}{\sqrt{\pi}}\right)$	$\frac{\rho}{8}\left(+\frac{\alpha W}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}+\frac{\alpha V}{\sqrt{\pi}}\right)$	$\frac{\rho}{8}\left(\frac{5}{2}\alpha^2 V+\frac{3\alpha^3}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}(W+U)\right)$
B_y	+	+ + +	+	+ − −	+ − +
C_y	−	− − +	−	− − +	− − −
D_y	−	− − +	−	− + −	− + −
E_y	+	+ − −	+	+ + +	+ + −
F_y	+	+ − −	+	+ − −	+ − −
G_y	−	− + −	−	− − +	− − +
H_y	−	− + −	−	− + −	− + +
A_z	$\frac{\rho}{8}\left(W+\frac{\alpha}{\sqrt{\pi}}\right)$	$\frac{\rho}{8}\left(+\frac{\alpha U}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}+\frac{\alpha W}{\sqrt{\pi}}\right)$	$\frac{\rho}{8}\left(+\frac{\alpha V}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}+\frac{\alpha W}{\sqrt{\pi}}\right)$	$\frac{\rho}{8}\left(\frac{\alpha^2}{2}+\frac{2\alpha W}{\sqrt{\pi}}\right)$	$\frac{\rho}{8}\left(\frac{5}{2}\alpha^2 W+\frac{3\alpha^3}{\sqrt{\pi}}+\frac{\alpha^2}{\pi}(U+V)\right)$
B_z	−	− − +	− − +	−	− − −
C_z	+	+ + +	+ − −	+	+ + −
D_z	−	− − +	− + −	−	− − +
E_z	+	+ − −	+ + +	+	+ − +
F_z	−	− + −	− − +	−	− + −
G_z	+	+ − −	+ − −	+	+ − −
H_z	+	− + −	− + −	−	− + +

In reading Table XX. it is to be understood that the several terms for B, C, D, E, F, G, H are the same as for A, with the exception of such changes of signs as are indicated in the table.

Terms of the order U^2 and U^3 have been omitted as being too small to affect the results of the investigation.

SECTION VII.—THE MEAN RANGE.

75. So far the gas has been supposed to be in uniform condition as regards space as well as time. When the condition varies from point to point, the results given in Table XX. will not hold good, for the condition of the molecules arriving from any particular direction which cross a plane at a point A will not be determined by the mean condition of the gas at A, but rather by the mean condition at the points at which the molecules receive the direction and velocity with which they cross the plane.

These points will not necessarily be the points at which the molecules last undergo encounter before crossing the plane, for one encounter may not be sufficient completely to modify their motion. In order, therefore, to determine from first principles the manner in which the molecules approach the point A, we must know the law of action between the molecules, and even then the complete solution would present difficulties which appear to be insuperable.

Fortunately, however, for the purposes of the present investigation a complete solution is not necessary. The point that has mainly to be considered is the effect of a solid surface on the mean condition of the molecules which cross a plane in its immediate neighbourhood. And the principal question is not how far such an effect would extend into gas in a particular condition, but what would be the nature of the effect at points to which it does extend, and what would be the comparative range of similar effects in gases the condition of which differ with respect to density and variation of temperature? If it should be found that the number and mean condition of the molecules which arrive at A from a given direction partake in a definite manner of the condition of the gas at a point in that direction whose distance s from A is a definite function of the density of the gas and some function of the temperature; such a solution would be sufficient to allow of the deduction of results corresponding to the experimental results.

Now it appears to follow from the view propounded at the commencement of this article, that in the interior of the gas there *must* be some distance s from a point A at which the mean condition of the gas must represent the mean condition of the molecules which reach A from that direction. This language is somewhat vague, but so must be the first idea. On closer inspection the question naturally arises as to what is meant by the mean condition of the gas, and by the mean condition of the molecules which reach A? Nor does this question at first sight appear to be difficult to answer. The mean condition of the gas appears most naturally to resolve itself into that which we can measure—the density ρ, the mean pressure $\frac{\rho}{3}(\overline{u^2}+\overline{v^2}+\overline{w^2})$ and the mean component velocities $\overline{u}, \overline{v}, \overline{w}$; and with respect to the mean condition of the arriving molecules why should not this be measured by their density, their mean energy and their mean component velocities? On comparing these with the corresponding quantities for the gas just mentioned, one point of doubt presents itself:

in the mean component velocities of the molecules arriving from one direction we have a very different thing from the mean component velocities of the gas. However, ignoring this caution, the most obvious supposition appears to be that as an approximation towards the condition of the molecules as they arrive at A, we may suppose them to come from a *uniform* gas having the density, mean pressure and component velocities of the gas at a point distant s from A in the direction from which they arrive. Such an assumption can be worked out, and the results compared with known experimental results. But we need go no farther than the case of gas at equal pressure and varying temperature. As applied to this case, our supposition leads to the inevitable conclusion that, unless s is zero, such a gas must be in motion from the colder to the hoter part with a velocity greater than its actual velocity, whatever this may be, which is absurd. This brings us back to the caution already mentioned respecting the difference between the component velocities of the group of molecules approaching A, and the component velocities of the gas. Without attempting to investigate this difference from first principles, we may follow the obvious course of attributing certain arbitrary mean component velocities to the uniform gas as from which the molecules are supposed to arrive at A.

We now suppose the molecules to arrive at A as from a uniform gas having the mean pressure and density at a distance s as before, but having arbitrary component velocities U, V, W (where U, V, W are so small that their squares may be neglected). This gets over the difficulty in the case mentioned above, for U, V, W being arbitrary can be so determined that the gas resulting from all the groups arriving at A shall have any mean velocity, and hence the mean velocity of the gas. It is only one such case, however, that we can meet in this way; for having once determined U, V, W, they are no longer arbitrary, and hence if the calculated results fit, to the same degree of approximation, all other cases, it must be that the approximation is a true one.

This test, however, can only be partially applied. As worked out in the subsequent sections of this paper, it was found that the supposition explained the phenomena of the radiometer and suggested the laws of transpiration and thermal transpiration exactly as they were afterwards realised. And in so far as they can be compared there is a complete agreement between the theoretical and experimental results.

Under these circumstances, the course which I first adopted in drawing up this paper was to found the theoretical investigation on such an assumption as has just been discussed.

The only other course was to look to first principles for the evidence wanting to establish the truth of the assumption. This I had attempted.

Obviously the first step in this direction was to examine the values of U, V, W as determined by the case of gas at varying temperature and uniform pressure. This showed *that if a plane be supposed to be moving through the gas with velocities U, V, W, then, measured with respect to the moving plane, the aggregate momenta carried from opposite sides across the plane are equal.*

This fact appeared pointed, but the exact point of it was not at once obvious, nor did it fully occur to me until I had completed the investigation founded as already described on the assumption.

Subsequently, however, working at the subject from the other end, so to speak, I came to see that whatever might be the action between the molecules, the probable effect of encounters in a varying gas would *not* tend to reduce the molecules after encounter to the same state as those of a *uniform gas* moving with the *mean component velocities of the varying gas,* but to a *uniform gas* moving with the halves of the mean component velocities *of all the molecules which cross a unit of surface in a unit of time—which pass through an element in a unit of time.*

I had not till then apprehended, nor do I know that it has anywhere been pointed out, that the mean component velocities of the molecules which pass through an element in a given time are not in the case of a varying gas, as they would be in that of a uniform gas (neglecting the squares of the mean component velocities), the doubles of the component velocities of the gas. But it turns out to be so (see Art. 77). And what is more, these mean component velocities are the very velocities U, V, W, which had been found to be necessary as already described.

The recognition of this fact therefore removed all fundamental difficulty as regarded the velocities U, V, W.

There still, however, remained the question as to whether the molecules might be considered to arrive in all respects to the same degree of approximation as from the same uniform gas—whether the molecules would arrive in respect to density from the same uniform gas as in respect to mean velocity, &c.; or whether severally in respect of density, mean velocity, &c., the uniform gas would correspond to different values of s? The answer to this question depends on the law of action between the molecules, and hence it is of necessity left for such light as accrues from the experiments and other known properties of gas.

It is, however, now proved (not altogether from first principles, but on certain elementary assumptions which might, it is thought, be deduced from first principles) that as regards number the molecules will arrive at A as from a uniform gas having the density, mean pressure and U, V, W, of the actual gas at a certain distance s from A, and that as regards mean velocity, mean square of velocity, and mean cube of velocity, the molecules will arrive as from uniform gas corresponding in each respect with the same or another point.

So that instead of having one value of s there are four; the numerical relations between which have not been determined from the elementary assumptions, but which are all shown to be functions of the temperature and inversely proportional to the density, and when the gas varies continuously independent of the direction from which the molecules arrive.

On comparing the theoretical results with those of experiment it is found—

1. That the values of s for density and mean square of velocity are equal;

2. That s for the mean cube does not enter into any of the experimental results of this investigation;

3. That s for the mean velocity has a real value, but there are no data for effecting a numerical comparison between this and the other value of s.

As this foundation of the theory on elementary assumptions renders it more satisfactory, it is introduced at length into this section of the paper. The argument, which is long and occupies Arts. (79 to 84), may be sketched as follows:—

Sketch of the method by which the fundamental theorems are deduced.

76. Upon certain elementary assumptions, which do not involve any particular law of action between the molecules, it is first shown that, in respect of density, mean velocity, &c., considered separately, any group of molecules whose directions of approach differ by less than a given small angle from any given direction BA, will enter the element at A (within a sufficient degree of approximation) *as if the gas were uniform* and had the same density and mean pressure as at B, and had mean component velocities which, although not the mean component velocities at B, are equal to one half the mean component velocities of all the molecules which enter an indefinitely small element at B in a unit of time. These component velocities, which are written U, V, W, cannot in the first instance be expressed in terms of known quantities, but they are shown to be functions of the position of B in the gas.

The distance AB or s is shown to be a function of the pressure and density of the gas, which function, although not completely expressed, as such an expression would involve the law of action between the molecules, is shown to be approximately independent of the variation of the density and pressure, and hence of the direction of AB.

The relations between ρ, α, U, V, W for a uniform gas may thus be used to express *severally* the density, mean velocity, &c., for each elementary group of molecules arriving at A. And since ρ, α, U, V, W are functions of the position of the point B (if x y z are the coordinates of A, and l m n are the direction cosines of AB) they are functions of $x+ls$, $y+ms$, $z+ns$, s having the value for the particular quantity to be represented. Therefore ρ, α, U, V, W for B may, by expansion, be represented by ρ, α, U, V, W for A, and their differential coefficients multiplied by powers of s. Thus the density, mean velocity, &c., of the molecules of each group arriving at A may *severally* be expressed in terms of ρ, α, U, V, W at A, and their differential coefficients multiplied by a particular value of s.

Therefore as the elementary portions of $\sigma(\mathrm{Q})$ for the group can always be expressed in terms of the density, mean velocity, &c., and l, m, n, it can be expressed in terms of ρ, α, U, V, W, for A, their differential coefficients multiplied by certain values of s and l, m, n. And, since all these quantities but l, m, n are independent of the direction of the group, by integrating for all values of l, m, n, $\sigma(\mathrm{Q})$ is found in terms of ρ, α, U, V, W, for A, and their differential coefficients multiplied by s.

It also appears that within the limits of the necessary approximation, terms multiplied by U^2, V^2, W^2, or differentials of the second order, may be neglected; so that $\sigma(Q)$ is expressed in terms of ρ, α, U, V, W, and their differential coefficients of the first order multiplied by some one or other of the several values of s.

U, V, W, are then at once found by putting $Q=M$, so that $\sigma_x(M)$, $\sigma_y(M)$, and $\sigma_z(M)$, are respectively $\bar{u}$, $\bar{v}$, and $\bar{w}$, which form the left sides of three equations (48) in which U, V, and W respectively appear on the right side.

It is difficult to give an intelligible sketch of so complicated a series of operations, but what has been stated above may serve to indicate the general scheme of this section.

Mean component velocities of the molecules which pass through an element.

77. It has been already pointed out that when the condition of the gas varies, the mean component velocities of all the molecules which in a unit of time pass through an element are *not*, to the same degree of approximation as they would be if the gas were uniform, the *doubles* of the *mean component velocities of the molecules in the element at the same instant.*

To express this, suppose that the condition of the gas varies only in the direction of x, so that the mean momentum in any direction perpendicular to x carried across all surfaces is zero.

Then taking a rectangular element, so that its edges are parallel to the axes, and its edges parallel to x are indefinitely short compared with its edges perpendicular to x, the only momentum carried through the element will be by molecules entering and leaving the faces perpendicular to x; and since the condition of the element remains unchanged the aggregate momentum of the molecules which enter must be equal to the aggregate momentum of the molecules which leave.

The aggregate momentum which enters at the face on the left is $\overset{u+}{\sigma_x}(Mu)$, or as it may be written $\overset{u+}{\Sigma}(Mu^2)$, while the aggregate momentum which enters on the right is $-\overset{u-}{\sigma_x}(Mu)$ or $-\overset{u-}{\Sigma}(Mu^2)$.

Therefore the whole momentum in the direction of x carried through the element in a unit of time is

$$\overset{u+}{\sigma_x}(Mu)-\overset{u-}{\sigma_x}(Mu) \quad \text{or} \quad \overset{u+}{\Sigma}(Mu^2)-\overset{u-}{\Sigma}(Mu^2)$$

And since the aggregate mass of the molecules which pass through the element in the same time is

$$\overset{u+}{\sigma_x}(M)-\overset{u-}{\sigma_x}(M) \quad \text{or} \quad \overset{u+}{\Sigma}(Mu)-\overset{u-}{\Sigma}(Mu)$$

the mean component velocity of all the molecules which pass through the element in a unit of time is

$$\frac{\overset{u+}{\sigma_x}(Mu)-\overset{u-}{\sigma_x}(Mu)}{\overset{u+}{\sigma_x}(M)-\overset{u-}{\sigma_x}(M)} \quad \text{or} \quad \frac{\overset{u+}{\Sigma}(Mu^2)-\overset{u-}{\Sigma}(Mu^2)}{\overset{u}{\Sigma}(Mu)-\overset{u-}{\Sigma}(Mu)}$$

which will not be, neglecting $\bar{u}^2$, the same as $2\bar{u}$, as it would be if the gas were uniform and moving with the velocity $\bar{u}$.

The same thing may be shown for faces parallel to x, and for variations in the directions y and z.

In all the phenomena considered, the velocity

$$\frac{\overset{u+}{\sigma_x}(\mathrm{M}u)-\overset{u-}{\sigma_x}(\mathrm{M}u)}{\overset{u+}{\sigma_x}(\mathrm{M})-\overset{u-}{\sigma_x}(\mathrm{M})}-\bar{u}$$

is very small compared with the mean velocity of a molecule; but the relation is of the same order as that of the unhindered rate of thermal transpiration and the mean velocity of a molecule.

78. The following limitations and definitions will tend to the simplification of subsequent expressions.

The condition of the gas.

All the assumptions and theorems, as indeed the entire investigation, with the exception of Arts. 108A and 109, relate to a simple gas in which the diameters of the molecules may be neglected in comparison with the mean distance which separates them, the condition of which gas is at all points steady as regards time, and the molecules of which are subjected to no external forces, such as gravity and electric attractions; and the term *gas* is to be understood in this sense unless otherwise defined.

The small quantities neglected.

As a first approximation, *i.e.*, in theorems (I.) and (II.) no account is taken of variations of the second order, such as are expressed by

$$\frac{d^2\rho}{dx^2}, \frac{d^2\alpha}{dx^2}, \text{\&c.}$$

the effects of such variations being too small to make any difference in the results of the first approximation.

Also throughout the investigation the velocity of the gas is assumed to be so small that such quantities as $\bar{u}^2$ and $\left(\frac{\overset{u+}{\sigma_x}(\mathrm{M}u)-\overset{u-}{\sigma_x}(\mathrm{M}u)}{\overset{u+}{\sigma_x}(\mathrm{M})-\overset{u-}{\sigma_x}(\mathrm{M})}\right)^2$ may be neglected.

Definitions.

An elementary group of molecules.—In addition to the separation of the molecules into the groups A, B, C, D, E, F, G, H, as explained in Art. 67, a further subdivision is necessary in order to render the reasoning of this section definite.

From any one of the eight groups are selected all the molecules having directions of motion which differ by less than certain small angles from a given direction, or, in other words, those molecules of which the directions of motion are parallel to some line which may be included within a pyramidal surface having indefinitely small angles at the apex. Such a group will be called *an elementary group*, and in this sense only will the term elementary group be used. The mean ray or axis of the pyramid is the mean direction of the group. And it is to be noticed that only those molecules that are moving in the *same direction* parallel to the axis of the pyramid are included in the *same group*, those with *opposite motion* constituting *another* elementary group.

The distinguishing features of an elementary group, apart from the direction of the group, are the number of molecules at any instant in a unit of volume—the symbol N will be used to signify this number; their mean velocity, mean square of velocity, &c., will be indicated without regard to direction by the symbols $\overline{\mathrm{V}}$, $\overline{\mathrm{V}^2}$; and to avoid confusion, instead of using Q to indicate the two latter quantities the letter G will be used to represent severally N, $\overline{\mathrm{V}}$, $\overline{\mathrm{V}^2}$, &c.

The resultant uniform gas.—It has been already pointed out (Art. 75) that if the encounters within an element of volume resulted in the molecules leaving the element in the same manner as they would leave if the gas about and within the element were uniform, this uniform gas must have component velocities which are one-half the mean component velocities of all the molecules of the varying gas which in a unit of time pass through the element. This uniform gas, which would also have approximately the mean pressure and density of the actual gas in the element, is called *the resultant uniform gas* of the gas within the element. U, V, W are used to designate its component velocities, ρ to express its density, and $\frac{\rho\alpha^2}{2}$ to express its pressure. U, V, W are functions of $\bar{u}$, $\bar{v}$, $\bar{w}$ and of the variations of ρ, α, or, in other words, they are functions of the condition of the gas at the point considered, but they cannot be completely determined in the first stage of the investigation.

The inequalities in elementary groups.—All the elementary groups relating to a unit of volume in a varying gas are compared with corresponding elementary groups in the resultant uniform gas for the element, and the differences in respect of the density and velocities of the molecules are spoken of as the *inequalities* of the group. There are only four quantities in respect to which the groups can be compared, namely: the *numbers of molecules*, the *mean velocity*, the *mean square*, and the *mean cube* of the velocity; essentially, therefore, the differences in these constitute the inequalities of the gorup.

Thus, if G standing for N, $\overline{\mathrm{V}}$, $\overline{\mathrm{V}^2}$ or $\overline{\mathrm{V}^3}$ refers to an elementary group of the resultant uniform gas for an *indefinitely small element*, and G+I refers to the corresponding elementary group of the varying gas, then I represents the inequality in an elementary group at a point as compared with the resultant uniform gas at that point.

When the element has small but definite dimensions (δr) the inequalities of the elementary groups entering or leaving will be

$$\pm\frac{dG}{dr}\frac{\delta r}{2}+I\pm\frac{dI}{dr}\frac{\delta r}{2};$$

for the inequality has reference to the uniform gas at a point distant $\frac{\delta r}{2}$ from the point at which I represents the inequalities, and therefore the change in G+I must be added to or subtracted from the inequality, as the case may be.

79. The following assumptions may all be deduced from first principles, but the necessary reasoning is long, and it is thought that the assumptions are sufficiently obvious.

Assumptions.

I. *That the condition of the gas, as already defined* (Art. 78), *at any instant within an element of volume depends entirely on the numbers and component velocities of the molecules which, in a unit of time, enter at each part of the surface of the element; and hence if the molecules enter one element in exactly the same manner as the molecules enter a geometrically similar element, the condition of the gas within the elements must be similar.*

II. *That the number and component velocities of the molecules which leave each elementary portion of the surface of an element, depend only on the condition of the gas within the element and the manner in which the molecules enter;* and therefore by (I.) depend only on the manner in which molecules enter. Also since the gas immediately outside the element consists of the molecules entering and leaving, its condition depends only on the molecules entering. So that if molecules enter corresponding portions of the surfaces of two geometrically similar elements in exactly the same manner the gas about the elements must be exactly similar.

III. *That whatever be the nature of the action between the molecules, the effect of encounters within an element must always tend to produce or maintain the same relative motion amongst the molecules, which relative motion is that of a uniform gas;* and hence the encounters must render the manner in which the molecules leave the element, as compared with that in which they enter, more nearly similar to the manner of a uniform gas.

That is to say, if A, B, C, D, &c., be a series of geometrically similar spherical elements, and the gas about B is such that the molecules enter B in exactly the same manner as they leave the opposite sides of A, and the gas about C such that the molecules enter as they leave the opposite side of B and so on, the gas about each element being such that the molecules enter the element exactly as they leave the opposite side of the preceding element, then according to the assumption the gas about each element will be more nearly uniform than that about the preceding element, so that eventually about the n^{th} element the gas would be uniform, n being indefinitely great.

This may be expressed algebraically. Putting h for the number of encounters necessary to obliterate the inequalities in the groups which pass through A in a unit of time, h will be infinite, and as I is so small that it may be considered as taking no part in the distribution, the rate of distribution will depend on the number of encounters in a unit of volume, and on some function $f(\alpha)$ of α, $\frac{2\alpha}{\sqrt{\pi}}$ being the mean velocity of the molecules.

Therefore approximately

$$\frac{d\text{I}}{dh}=-f(\alpha)\text{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

So that if I′ is the initial value of I, then after h encounters we have integrating

$$\text{I}=\text{I}'e^{-f(\alpha)h},$$

and if h is infinite

$$\text{I}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$f(\alpha)$ is a positive function of α, and is not a function of I; but both as regards form and coefficients $f(\alpha)$ may depend on the nature of the quantity G.

The question whether $f(\alpha)$ is different for any or all of the quantities N, $\overline{\text{V}}$, $\overline{\text{V}^2}$, &c., must depend on the nature of the action between the molecules during encounters.

If therefore by comparing the mathematical results with those from experiments the several values of $f(\alpha)$ can be compared, a certain amount of light would be thrown on the action between the molecules. So far, however, the conditions of equilibrium in the interior of gas of which the temperature varies form the only instance in which the values of $f(\alpha)$ are brought into direct comparison. This instance affords means of comparing the values of $f(\alpha)$ for N and $\overline{\text{V}^2}$, and shows that these values must be equal. As regards $f(\alpha)$ for $\overline{\text{V}}$ or $\overline{\text{V}^3}$, there are no experimental results which furnish any further light than that $f(\alpha)$ has real positive values.

These questions do not rise in this investigation, since $f(\alpha)$ for $\overline{\text{V}^3}$ does not appear in the results, and should $f(\alpha)$ have a different value for $\overline{\text{V}}$ from that which it has for N and $\overline{\text{V}^2}$, the only result would be a numerical difference in certain coefficients as to the comparative value of which the experiment affords no approximate evidence.

IV. *That when the molecules which enter or leave an element of volume in a unit of time are considered separately, the proportion of the molecules* (N $\overline{\text{V}}$) *entering in a unit of time in each entering group which will subsequently undergo encounters within the element, and the proportion of the molecules leaving in a unit of time, in each leaving group, which have undergone encounters within the element, are approximately proportional to the mean distance* (δr) *through the element in direction of the group and to the number of molecules in each unit of volume of the element.*

V. *That the mean effect of encounters in distributing the several inequalities of the molecules which, entering in a unit of time, encounter within the element is a function* ($f(\alpha)$) *of the mean velocity of the molecules within the element at the instant.*

Fundamental Theorems.

80. On the assumptions I. to V., remembering the fact pointed out in Arts. 75 and 79 with respect to the component velocities of the resultant uniform gas, the following theorems are established:—

Theorem (I.).—*Each of the several inequalities, as defined in Art. 78, in every elementary group of molecules which in a unit of time leave an element of volume of small but definite size will severally be less than in the corresponding elementary group, which in the same time enter the element in the same direction by quantities which bear approximately the same relation to the mean inequalities of the two groups, as the distance through the element in direction of the group bears to a distance (s) which is a function of the density of the gas, and the mean square of the velocity of the molecules only.*

To express this theorem algebraically, let G and I, as explained in the last article, refer to the point in the middle of the element. Then the inequality in the entering group is expressed by

$$-\frac{d\mathrm{G}}{dr}\frac{\delta r}{2}+\mathrm{I}-\frac{d\mathrm{I}}{dr}\frac{\delta r}{2}$$

and for the leaving group by

$$\frac{d\mathrm{G}}{dr}\frac{\delta r}{2}+\mathrm{I}+\frac{d\mathrm{I}}{dr}\frac{\delta r}{2}$$

And what the theorem asserts is

$$\frac{d}{dr}(\mathrm{G}+\mathrm{I})\,\delta r=\frac{\delta r}{s}\mathrm{I}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\qquad(25)$$

wherein s is a function of ρ and α^2 only.

Proof of Theorem (I.).

(*a*) From assumptions I. and II., Art. 79, it follows at once that when the condition of the gas varies from point to point, the molecules cannot enter an element of volume in the same manner as they would from any uniform gas.

(*b*) From (*a*) and assumption III. it follows that the effect of encounters within an element in a varying gas is to render the manner in which the molecules leave as compared with that in which they enter more nearly similar to that of some uniform gas.

(*c*) The uniform gas referred to in (*b*) must, as has been already pointed out, have component velocities equal to half the mean component velocities of all the molecules which in a unit of time pass through the element.

This at once follows from the illustration appended to assumption III., Art. 79. For the molecules which leave an element in a unit of time must have the same mean component velocities as those which enter, their aggregate mass being the same and the momentum within the element remaining unaltered, and as the molecules enter each successive element in the same manner as they left the preceding, the molecules which enter the n^{th} element in a unit of time must have the same mean component velocities as those which enter the first; but in the n^{th} element the gas is uniform. Therefore, if U, V, W are the component velocities of the uniform gas, when these are small so that we may neglect U^2, V^2, W^2

$$U=\tfrac{1}{2}\frac{\overset{u+}{\sigma_x}(Mu)-\overset{u-}{\sigma_x}(Mu)}{\overset{u+}{\sigma_x}M-\overset{u-}{\sigma_x}(M)},\ V=\tfrac{1}{2}\frac{\overset{v+}{\sigma_y}(Mv)-\overset{v-}{\sigma_y}(Mv)}{\overset{v+}{\sigma_y}(M)-\overset{v-}{\sigma_y}(M)},\ W=\tfrac{1}{2}\frac{\overset{w+}{\sigma_z}(Mw)-\overset{w-}{\sigma_z}(Mw)}{\overset{w+}{\sigma_z}M-\overset{w-}{\sigma_z}M}\quad . \ . \ . \ (26)$$

The number of molecules which enter the n^{th} element will also be equal to the number which enter the 1^{st}.

Therefore putting $\overline{w^2}+\overline{v^2}+\overline{u^2}=\frac{3\alpha^2}{2}$, and using the dash to indicate the first element

$$\rho\alpha=\rho'\alpha' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (27)$$

And the energy carried into the n^{th} is equal to the energy carried into the first element. Therefore

$$\rho\alpha^3=\rho'\alpha'^3 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (28)$$

From which equations

$$\alpha^2=\alpha'^2 \text{ and } \rho=\rho' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (29)$$

Or the density and pressure of the uniform gas is approximately the same as the density and mean pressure of the actual gas. This uniform gas is, therefore, the resultant uniform gas according to the definition Art. 78.

(*d*) From assumptions IV. and V. it follows directly that the several changes in the inequalities considered separately of each elementary group which enters the element in a unit of time will be proportional to the mean inequalities of the group as it enters and leaves multiplied by $f(\alpha)$ and by the product of $\frac{\rho}{M}$ and the mean distance through the element traversed by the group.

Or, as before, putting I for the mean inequality of the group as it enters and leaves in respect of G, the separate inequalities are

$$-\frac{d}{dr}(G+I)\ \frac{\delta r}{2}+I \text{ and } \frac{d}{dr}(G+I)\frac{\delta r}{2}+I.$$

Whence from assumptions IV. and V. it follows that

$$\frac{d}{dr}(G+I)\delta r=f(\alpha)\frac{\rho}{M}\delta r I \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (30)$$

And from the dimensions of this equation it follows that $\frac{\mathrm{M}}{f(\alpha)\rho}$ represents a distance. Therefore putting s for this distance

$$\frac{d}{dr}(\mathrm{G}+\mathrm{I})\delta r=\frac{\delta r}{s}\mathrm{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

[Q. E. D.]

Corollary to Theorem (I.).

When $\frac{d\mathrm{G}}{dr}$ is nearly constant, so that we may neglect $s\frac{d^2\mathrm{G}}{dr^2}$ as compared with $\frac{d\mathrm{G}}{dr}$, then integrating equation (31) we have

$$\left.\begin{array}{c} \mathrm{I}=s\frac{d\mathrm{G}}{dr}+\mathrm{C}e^{-\frac{r}{s}} \\ \text{or} \\ \frac{d\mathrm{G}}{dr}=\frac{\mathrm{I}}{s}-\frac{\mathrm{C}}{s}e^{-\frac{r}{s}} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Near a solid surface.

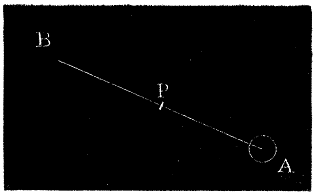
Equation (32) shows the nature of the inequalities as affected by discontinuity such as may arise at a solid surface. The last term on the right gives the effect of discontinuity for an element at a distance r from the surface, r being measured in the direction of the group. This effect diminishes as r increases.

In the first of equations (32) we may obviously put $s_1\frac{d\mathrm{G}}{dr}$ for $s\frac{d\mathrm{G}}{dr}+\mathrm{C}e^{-\frac{r}{s}}$, s_1 being a function of the position of the element and of the direction of the group.

Theorem (II.).—*When the variation in the condition of the gas is approximately constant, then in respect of any one of the quantities* N, $\bar{\mathrm{v}}$, $\overline{\mathrm{v}^2}$, *&c., each elementary group of molecules entering a small element of volume at any point will enter approximately as if from the resultant uniform gas at a point in the direction from which the group arrives, the distance of which point from the element is a function of the mean velocity of a molecule and inversely proportional to the number of molecules within a unit of volume, and is independent of the variation of the gas and the direction of the elementary group.*

To illustrate this, supposing a small spherical element at A, and considering the group arriving in the direction BA, then if the gas varies in the direction BA the resultant uniform gas for points along BA will differ, and if A were to be surrounded

Fig. 11.



by a gas identical with the resultant uniform gas at a point P, the elementary group in the direction BA or PA would arrive at the element with different values as to density, mean velocity, &c., from a similar group if the gas were identical with the resultant uniform gas at another point in AB.

Now what the theorem asserts is, that there is some point P_1 at which the *resultant uniform gas* is such that the elementary group in direction BA would arrive with approximately the same value of N as the actual group, and that the distance P_1A is independent of the direction of BA, *i.e.*, would be the same for all directions from A. In the same way there is some point P_2 at which the resultant uniform gas is such that the group of molecules B A would have the same value of $\bar{v}$ as for the actual group, and so for $\overline{v^2}$ and $\overline{v^3}$.

It is not however asserted that AP_1, AP_2, &c., either are or are not identical.

Proof of Theorem (II.).

This follows directly from theorem (I.).

Taking a series of elements bounded by a cylindrical surface described about the element at A and having its axis in the direction of the group, then all the molecules of the group leaving one element may be supposed to enter the next.

In entering the first element at B there will be a difference I between the value of G for the actual group and the value of G for the resultant uniform gas. If G_B is taken for the resultant uniform gas G_B+I_B will represent the corresponding value for the actual gas at B.

On emerging from the first element G+I for the group will, by theorem (I.), have been diminished by $\frac{\delta r}{s}I$, δr being the thickness of the element, on emerging from the next element, G+I will be still further diminished by $\frac{\delta r}{s}I$, and so on through all the elements, the total diminution of G+I being equal to—

$$\int_0^s \frac{I}{s} dr.$$

And by the corollary to theorem (I.), since the variation in the condition of the gas is approximately constant, I is approximately constant through the distance s, and s will be approximately constant through this distance; therefore

$$\int_0^s \frac{I}{s} dr = I_B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Hence, having traversed the distance s, the group will emerge having

$$\begin{aligned} G+I &= G_B+I_B-I_B \\ &= G_B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34) \end{aligned}$$

That is, on arriving at A, the molecules will, in respect of G, enter the element as if from the resultant uniform gas at B, a point in the direction of the group, the distance s of which from A is a function of α, is inversely proportional to $\frac{\rho}{M}$ and is independent of the variation of the gas and of the direction of the group. [Q. E. D.]

Corollary to Theorem (II.).—The effect of a solid surface.

If in the neighbourhood of A there is a solid surface such that, if B is a point on this surface, BA is of the same order of magnitude as s, then putting r=BA for the group arriving at A from the direction BA, equation (33) gives

$$G_A+I_A=G_B+I_B-\int_0^r\frac{I}{s}dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and substituting for $\frac{I}{s}$ from equation (32) and integrating

$$G_A+I_A=G_B+I_B-r\frac{dG}{dr}-C(e^{-\frac{r}{s}}+1),$$

or since $G_B=G_A+r\frac{dG}{dr}$ and $I_B-C=s\frac{dG}{dr}$, therefore

$$I_A=s\frac{dG}{dr}-Ce^{-\frac{r}{s}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

C will be a function of l, m, n, and it may be written $f(lmn)s\frac{dG}{dr}$; therefore

$$I_A=s\frac{dG}{dr}\{1-f(lmn)e^{-\frac{r}{s}}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The mean range.

81. The distance s, or $\frac{M}{f(\alpha)\rho}$ (equation 30) is thus shown to be the distance at which the elementary groups radiating outwards from a point have the mean value of G for the molecules which, in a unit of time, pass the central point. And hence it is proposed to call s the *mean range of the quantity* G.

The mean range is thus seen to be approximately independent of the space variations of the gas, but since s involves $f(\alpha)$, which, as pointed out in assumption III., Art. 78, may, so far as is yet known, have different values for $\overline{V}$ and $\overline{V^3}$ from its values for N and $\overline{V^2}$, which latter are equal, so the values of s for the mean velocity and mean cube of the velocity may be different from the values of s for the density and mean square of the velocity, which latter are equal.

Such a difference in the values of s, however, is not important as regards the immediate results of this investigation, and in the absence of any evidence to the contrary all values of s will be treated as equal.

The mean component values of s and general expression of σ(Q).—Gas continuous.

82. When the gas is continuous, by theorem (II.) all values of $\sigma(Q)$ for the groups A, B, C, &c., at any point may severally be expressed as functions of ρ, α, U, V, W for this point, their space variations, and s.

The first step is to express as a function of ρ, α, U, V, W, the elementary portion of $\sigma(Q)$ belonging to an elementary group of molecules, and then to integrate for the larger groups.

Putting $\sigma'(Q)$ for the value of $\sigma(Q)$, which would result from the resultant uniform gas at a point, and $\delta\sigma(Q)$ for the elementary portion of $\sigma(Q)$ belonging to an elementary group whose directions are l, m, n, then since ρ, α, U, V, W, for any point x, y, z, are functions of x, y, z, $\delta\sigma(Q)$ is a function of x, y, z, and for the point $x+ls$, $y+ms$, $z+ns$ we have

$$\delta\sigma(Q)=\delta\sigma'(Q)+s\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\delta\sigma'(Q) \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

together with terms which are neglected as small.

Whence integrating for all the groups in A, and putting A for $\sigma'(Q)$

$$\sigma^a(Q)=A+s\int_0^\pi\int_0^{\frac{\pi}{2}}\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\delta A \sin\theta d\theta d\phi \quad . \quad . \quad . \quad . \quad . \qquad (39)$$

where $\cos\theta=l$, $m=\sin\theta\cos\phi$, $n=\sin\theta\sin\phi$, and similarly for the other seven groups, B, C, &c.

The values of A, &c., are given in Table XX.

The integrals of $s\left\{\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\delta A\right\}\sin\theta d\theta d\phi$ will involve terms which will be the differentials of the corresponding terms in Table XX., multiplied by s and by certain numerical coefficients which are the mean values of l, of l^2 divided by l, and so on, and which may be written $\bar{l}$, $\frac{\overline{l^2}}{\bar{l}}$, $\frac{\overline{lm}}{\bar{l}}$, &c. The values of s multiplied by these coefficients are the mean component values of s for ρ, α, α^2, &c., for the groups A, B, C, &c. As it is these component values which come into comparison with the distances from a solid surface, it is important to preserve the coefficients, therefore instead of using the numerical values they will be indicated by the letters L_1, L_2, &c., as about to be assigned.

Putting i for unity or any power of α or U, V, W, the coefficients by which the differentials of the corresponding terms in Table XX. must be multiplied, are as follows:—

$$\left.\begin{array}{llll}
\frac{d\overline{\rho i}}{dx}, \text{ \&c.} & \text{by} & \overline{l}=\frac{1}{2} & \text{expressed by } L_1, \\
\frac{d\overline{\rho\xi i}}{dx}, \frac{d\overline{\rho\eta i}}{dy}, \frac{d\overline{\rho\zeta i}}{dz} & ,, & \frac{\overline{l^2}}{\overline{l}}=\frac{2}{3} & \qquad ,, \qquad L_2, \\
\frac{d\overline{\rho\xi i}}{dy}, \frac{d\overline{\rho\xi i}}{dz}, \text{ \&c.} & ,, & \frac{\overline{lm}}{\overline{l}}=\frac{4}{3\pi} & \qquad ,, \qquad L_3, \\
\frac{d\overline{\rho\xi^2 i}}{dx}, \frac{d\overline{\rho\eta^2 i}}{dy}, \frac{d\overline{\rho\zeta^2 i}}{dz} & ,, & \frac{\overline{l^3}}{\overline{l^2}}=\frac{3}{4} & \qquad ,, \qquad L_4, \\
\frac{d\overline{\rho\xi^2 i}}{dy}, \frac{d\overline{\rho\xi^2 i}}{dz}, \text{ \&c.} & ,, & \frac{\overline{l^2m}}{\overline{l^2}}=\frac{3}{8} & \qquad ,, \qquad L_5, \\
\frac{d\overline{\rho\xi\eta i}}{dx}, \frac{d\overline{\rho\xi\zeta i}}{dx}, \text{ \&c.} & ,, & \frac{\overline{l^2m}}{\overline{lm}}=\frac{3}{16}\pi & \qquad ,, \qquad L_6.
\end{array}\right\} \quad \ldots\ldots \quad (40)$$

The coefficients L_1, L_2, &c., all occur in some one or other of the expressions for A, B, C, &c.; but when these expressions come to be added together it is found that L_2 is the only coefficient of s which has to be considered. This being the case, instead of L_2s, the simple s will be used, so that in all subsequent expressions

$$s=\frac{2}{3}(\text{mean range}) \quad \ldots\ldots\ldots\ldots \quad (41)$$

The signs of the products of s and the differentials of the several terms in the table may, as will be seen from Art. 69, be expressed in the following manner, ignoring the numerical coefficients L_1, L_2, &c.

$$\left.\begin{array}{l}
\sigma_x(Q)=A_x+B_x+C_x+D_x+E_x+F_x+G_x+H_x \\
\qquad -s\frac{d}{dx}(A_x+B_x+C_x+D_x-E_x-F_x-G_x-H_x) \\
\qquad -s\frac{d}{dy}(A_x+B_x-C_x-D_x+E_x+F_x-G_x-H_x) \\
\qquad -s\frac{d}{dz}(A_x-B_x+C_x-D_x+E_x-F_x+G_x-H_x),
\end{array}\right\} \quad . \quad (42)$$

with similar expressions for $\sigma_y(Q)$ and $\sigma_z(Q)$, the suffix to the letters being the same as the suffix to $\sigma(Q)$ on the left.

The following are the resulting values of $\sigma(Q)$ which are required for this investigation, terms of the order U^2 having been neglected.

$$\left.\begin{aligned}\sigma_x(\mathrm{M})&=\rho\mathrm{U}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha}{dx},\\ \sigma_y(\mathrm{M})&=\rho\mathrm{V}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha}{dy},\\ \sigma_z(\mathrm{M})&=\rho\mathrm{W}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha}{dz},\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(43)$$

$$\left.\begin{aligned}\overset{u+}{\sigma}_x(\mathrm{M}u)&=\frac{\rho\alpha^2}{4}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dx}-\frac{s}{4}\frac{d\rho\alpha^2}{dx},\\ \overset{u-}{\sigma}_x(\mathrm{M}u)&=\frac{\rho\alpha^2}{4}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dx}+\frac{s}{4}\frac{d\rho\alpha^2}{dx},\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(44)$$

$$\left.\begin{aligned}\overset{v+}{\sigma}_y(\mathrm{M}u)&=+\frac{\rho\alpha\mathrm{U}}{2\sqrt{\pi}}-\frac{s}{2\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dy}-\frac{s}{2\pi}\frac{d\rho\alpha^2}{dx}-\frac{s}{2\sqrt{\pi}}\frac{d\rho\alpha\mathrm{V}}{dx},\\ \overset{v-}{\sigma}_y(\mathrm{M}u)&=-\frac{\rho\alpha\mathrm{U}}{2\sqrt{\pi}}-\frac{s}{2\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dy}+\frac{s}{2\pi}\frac{d\rho\alpha^2}{dx}-\frac{s}{2\sqrt{\pi}}\frac{d\rho\alpha\mathrm{V}}{dx},\end{aligned}\right\}\quad\ldots\quad(45)$$

$$\left.\begin{aligned}\sigma_x(\mathrm{M}u)&=\frac{\rho\alpha^2}{2}-\frac{2s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dx},\\ \sigma_y(\mathrm{M}u)&=-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{V}}{dx}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dy},\\ \sigma_z(\mathrm{M}u)&=-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{W}}{dx}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha\mathrm{U}}{dz},\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(46)$$

with corresponding equation for $\mathrm{Q}=\mathrm{M}v$, $\mathrm{Q}=\mathrm{M}w$,

$$\sigma_x\{\mathrm{M}(u^2+v^2+w^2)\}=\tfrac{5}{2}\rho\alpha^2\mathrm{U}-\frac{3s}{\sqrt{\pi}}\frac{d\rho\alpha^3}{dx}\quad\ldots\ldots\quad(47)$$

and similar equations.

The values of U, V, W.

Hitherto U, V, W have been treated merely as functions of x, y, z. They are, however, completely expressed by equation (43).

For remembering that $\sigma_x(\mathrm{M})$, $\sigma_y(\mathrm{M})$, $\sigma_z(\mathrm{M})$ are respectively equivalent to $\rho\bar{u}$, $\rho\bar{v}$, $\rho\bar{w}$, we have

$$\left.\begin{aligned}\mathrm{U}&=\bar{u}+\frac{s}{\sqrt{\pi}\rho}\frac{d\rho\alpha}{dx},\\ \mathrm{V}&=\bar{v}+\frac{s}{\sqrt{\pi}\rho}\frac{d\rho\alpha}{dy},\\ \mathrm{W}&=\bar{w}+\frac{s}{\sqrt{\pi}\rho}\frac{d\rho\alpha}{dz}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(48)$$

The neighbourhood of a solid surface.

83. In this case we have by the corollary to theorem (II.)

$$\delta\sigma(\mathrm{Q})=\delta\sigma'(\mathrm{Q})+s\{1-f(lmn)e^{-\frac{r}{s}}\}\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\delta\sigma'(\mathrm{Q}) \quad . \quad . \quad . \quad (49)$$

or as in equation (39)

$$\overset{a}{\sigma}(\mathrm{Q})=\mathrm{A}+s\int_0^{\pi}\int_0^{\frac{\pi}{2}}\{1-f(lmn)e^{-\frac{r}{s}}\}\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)\delta\mathrm{A}\sin\theta d\theta d\phi \quad . \quad . \quad . \quad (50)$$

In this case it is clear that the coefficients which correspond to L_1, L_2, &c., Art. 82, will be functions of the position of the point with respect to the solid surface, and will depend on the value of $f(lmn)$. $f(lmn)$ will obviously depend to some extent on the action between the molecules and the solid surface. It appears, however, that when the solid surface extends in all directions in its own plane to distances which are great as compared with s, and the variation Q is perpendicular to this plane, the result of the integration of equation (50) is the same as that of equation (39). For taking the solid surface parallel to xy

$$\sigma'(\mathrm{Q})=\overset{w+}{\sigma}(\mathrm{Q})+\overset{w-}{\sigma}(\mathrm{Q}),$$

and by symmetry, since Q varies only in the direction z, for two opposite groups such as a, b

$$\begin{aligned}\overset{a}{\sigma}(\mathrm{Q})+\overset{b}{\sigma}(\mathrm{Q})&=\overset{a}{\sigma}'(\mathrm{Q})+\overset{b}{\sigma}'(\mathrm{Q})\\&=\mathrm{A}+\mathrm{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)\end{aligned}$$

Therefore the integral of the last term of equation (50) for A will have the same value but the opposite sign as for B. Hence since the solid surface can only be on one side of the element, say the side A C G E, fig. 10, Art. 66, and $\frac{r}{s}$ will be infinite for the group B, or for this group equation 50 is identical with equation 39, therefore for either of the opposite groups the results of the integration of (50) are the same as of (39).

Near the edge of a solid surface, or when Q varies in some direction parallel to the surface, equation (51) no longer holds good, and then the coefficients corresponding to L_1, L_2, &c., will depend on the position of the element with respect to the solid surface and on the action between the molecules and the solid surface.

In dealing with such cases two courses were open—the one was to try and find some form for $f(lmn)$ which would satisfy the equations, the other course, and that which is here adopted, is to introduce arbitrary functions s_1, s_2, in place of s, and subsequently to determine the form of s_1, s_2, so as to satisfy the experimental results.

84. The only case of importance in this investigation is that in which the temperature varies along a solid surface and is constant at right angles.

Taking $z=-c$ as the equation to the solid surface, and supposing the gas uniform in the direction y, and that $\frac{d\alpha}{dz}=\frac{d\rho}{dz}=0$, then if xyz are the coordinates of a point P and z is greater than $-c$ the effect of the solid surface will be to alter the values of s in the terms involving the differentials of ρ and α. Using a suffix to indicate that the values of s for such terms is arbitrary, we may proceed to determine the values of $\sigma(\mathrm{Q})$, as in Art. 82. The important cases are $\mathrm{Q}=\mathrm{M}u$ and $\mathrm{Q}=\mathrm{M}$.

Remembering that s_1 refers to such terms in A C E G as involve $\frac{d\alpha}{dx}$ or $\frac{d\rho}{dx}$, and that $\mathrm{W}=0$, we have by the method of Art. 82

$$\sigma_z(\mathrm{M}u)=\frac{s-s_1}{2\pi}\frac{d\rho\alpha^2}{dx}-\frac{s}{\sqrt{\pi}}\rho\alpha\frac{d\mathrm{U}}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52)$$

$$\left.\begin{aligned}\sigma_x(\overset{w+}{\mathrm{M}})&=\tfrac{1}{2}\rho\mathrm{U}-\frac{s_1}{2\sqrt{\pi}}\frac{d\rho\alpha}{dx}-\frac{s\rho}{2}\frac{d\mathrm{U}}{dz}\\ \sigma_x(\overset{w-}{\mathrm{M}})&=\tfrac{1}{2}\rho\mathrm{U}-\frac{s}{2\sqrt{\pi}}\frac{d\rho\alpha}{dx}+\frac{s\rho}{2}\frac{d\mathrm{U}}{dz}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53)$$

Further, to adapt these equations to the form required, put $\bar{u}_1$ and $\bar{u}_2$ for the mean velocity of the opposite groups $w+$ and $w-$, so that $\sigma_x(\overset{w+}{\mathrm{M}})=\rho\frac{\bar{u}_1}{2}$, $\sigma_x(\overset{w-}{\mathrm{M}})=\rho\frac{\bar{u}_2}{2}$.

Then since $\bar{u}$ may be taken as constant in the direction of x, we have by corollary to theorem (II.) and equation (51)

$$\rho\frac{\bar{u}_1}{2}-\rho\frac{\bar{u}_2}{2}=-s\rho\frac{d\bar{u}}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53a)$$

Subtracting equations (53)

$$-s\rho\frac{d\bar{u}}{dz}=\frac{s-s_1}{2\sqrt{\pi}}\frac{d\rho\alpha}{dx}-s\rho\frac{d\mathrm{U}}{dz},$$

and substituting for $\frac{d\mathrm{U}}{dz}$ in equation (52)

$$\sigma_z(\mathrm{M}u)=\frac{s-s_1}{2\pi}\rho\alpha\frac{d\alpha}{dx}-\frac{s}{\sqrt{\pi}}\rho\alpha\frac{d\bar{u}}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (54)$$

If the point P lies between two surfaces, then putting s_2 as an arbitrary function we have

$$\sigma_z(\mathrm{M}u)=\frac{s_2-s_1}{2\pi}\rho\alpha\frac{d\alpha}{dx}-\frac{s}{\sqrt{\pi}}\rho\alpha\frac{d\bar{u}}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (54a)$$

For further consideration of s_2-s_1 see Art. 96.

[This section (VII.) was revised and somewhat enlarged in August, 1879, in accordance with a suggestion made by one of the Referees.]

SECTION VIII.—THE EQUATIONS OF STEADY MOTION.

85. If Q is a quantity of such a nature that ΣQ cannot change on account of any mutual action between the molecules within a unit of volume; and further, if we assume that the molecules within a unit of volume at any instant are not subject to any influence other than those which they exert on one another, then whatever change may take place in an elementary volume must be on account of the excess of Q carried into the unit of volume over and above that which is carried out; and we have

$$\frac{d\Sigma(Q)}{dt}=-\frac{d\sigma_x(Q)}{dx}-\frac{d\sigma_y(Q)}{dy}-\frac{d\sigma_z(Q)}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$\frac{d\Sigma(Q)}{dt}$ is the rate at which $\Sigma(Q)$ is increasing at a point fixed in space. Hence if the condition of the gas is steady

$$\frac{d\Sigma Q}{dt}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Therefore if the condition of the gas is steady, we have

$$\frac{d\sigma_x(Q)}{dx}+\frac{d\sigma_y(Q)}{dy}+\frac{d\sigma_z(Q)}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

86. If, therefore, we put $Q=M$, equation (57) gives us the condition of steady density.

Whereas if we put successively $Q=Mu$, $Q=Mv$, $Q=Mw$, we have from equation (57) the conditions of steady momentum in the directions of the axes.

And if we put $Q=M(u^2+v^2+w^2)$ we have the condition of steady pressure.

The condition that the gas may be subject to no distorsion or shear stress.

87. In order that $\sigma_x(Mv)$, $\sigma_x(Mw)$, $\sigma_y(Mw)$, $\sigma_y(Mu)$, $\sigma_z(Mu)$, and $\sigma_z(Mv)$ may respectively be zero for all positions of the axes, we must have

$$\sigma_x(Mu)=\sigma_y(Mv)=\sigma_z(Mw) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Therefore from the first of equations (46) and like equations

$$s\frac{d\rho\alpha U}{dx}=s\frac{d\rho\alpha V}{dx}=s\frac{d\rho\alpha W}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

These are the conditions that there shall be no tangential stress within the gas at a distance from a solid.

Coupled with the conditions for steady density, steady momentum, and steady pressure, these equations are, within the limits of our approximation, equivalent to

$$\frac{d^2\alpha^2}{dx^2}=\frac{d^2\alpha^2}{dy^2}=\frac{d^2\alpha^2}{dz^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

and

$$\frac{\rho\alpha^2}{2}=p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

where p the pressure is constant throughout the gas.

88. The important condition in this investigation is that the tangential force on a solid surface shall be zero.

This condition can only be obtained by the aid of some *assumption* as to the action between the molecules and the surface. An extremely obvious assumption will suffice, viz.: that the tangential force on the surface has the same *direction* as the momentum, parallel to the surface, of all the molecules which reach the surface in a unit of time.

The condition that there shall be no force on the surface is, then, that the momentum parallel to the surface which is carried up to the surface shall be zero.

Thus, if the axial planes be solid surfaces, we have from the values of $\overset{w-}{\sigma}_z(\mathrm{M}u)$, $\overset{w-}{\sigma}_z(\mathrm{M}v)$, &c., equations (45) that

$$\mathrm{U}=\mathrm{V}=\mathrm{W}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

at the surface.

If, further, there is no tangential stress within the gas, it appears from equations (59), (60), and (61), that equation (62) must hold throughout the gas.

The condition that there shall be no tangential stress on a particular solid surface, say, the plane of xy, is satisfied if at that surface $\rho\alpha^2$ is constant and

$$\mathrm{U}=0,\ \mathrm{V}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

and

$$\frac{d\mathrm{U}}{dz}=\frac{d\mathrm{V}}{dz}=\frac{d\mathrm{W}}{dx}=\frac{d\mathrm{W}}{dy}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

This appears at once from the values of $\overset{w-}{\sigma}_z(\mathrm{M}u)$, $\overset{w-}{\sigma}_z(\mathrm{M}v)$ obtained as equations (45).

[The revision of Section VII. necessitated certain alterations in Arts. 87 and 88; these articles were therefore revised in the proof, December, 1879.]

SECTION IX.—APPLICATION TO TRANSPIRATION THROUGH A TUBE.

89. It will be sufficient to consider the simplest cases; hence it is supposed that the gas is transpiring through a tube of uniform section, and further that the tube is of unlimited breadth, the surfaces being planes parallel to the plane $x\ y$; the axis of x is

taken for the axis of the tube, and it is assumed that all perpendicular sections of the tube are surfaces of equal pressure and temperature, the variation of temperature and pressure being in the direction x.

The equations to the surfaces of the tube are taken

$$z=\pm c \quad \ldots \ldots \ldots \quad (65)$$

90. From equation (57) we have for steady momentum

$$\frac{d}{dx}\{\sigma_x(\mathrm{M}u)\}+\frac{d}{dz}\{\sigma_z(\mathrm{M}u)\}=0 \quad \ldots \ldots \ldots \quad (66)$$

for steady density

$$\frac{d}{dx}\{\sigma_x(\mathrm{M})\}+\frac{d}{dz}\{\sigma_z(\mathrm{M})\}=0 \quad \ldots \ldots \ldots \quad (67)$$

and for steady pressure

$$\frac{d}{dx}\{\sigma_x\mathrm{M}(u^2+v^2+w^2)\}+\frac{d}{dz}\{\sigma_z\mathrm{M}(u^2+v^2+w^2)\}=0 \quad \ldots \ldots \quad (68)$$

Steady pressure not important.

91. In a tube, since heat may be communicated from the surface to the gas, the temperature may be maintained constant; and if the density be steady the pressure will also be steady, hence the condition of steady pressure ceases to be important. The law of variation of temperature is determined by the sides of the tube.

Transpiration when s is small as compared with c.

92. If s is so small that it is unnecessary to consider the layer of gas throughout which the direct influence arising from the discontinuity at the surface extends, substituting in equation (66) from equations (46), and putting $\frac{dp}{dx}=\frac{d}{dx}(\sigma_x(\mathrm{M}u))$, which we may do within the limits of our approximation, we have for steady momentum, since $\mathrm{W}=0$ in the tube

$$\frac{dp}{dx}=\frac{1}{\sqrt{\pi}}\frac{d}{dz}\left(s\frac{d\rho\alpha\mathrm{U}}{dz}\right) \quad \ldots \ldots \ldots \quad (69)$$

And from equations (43) and (67), since ρ and α do not vary across the tube, we have for steady density

$$\frac{1}{\sqrt{\pi}}\frac{d}{dx}\left(s\frac{d\rho\alpha}{dz}\right)=\frac{d}{dx}(\rho\mathrm{U}) \quad \ldots \ldots \ldots \quad (70)$$

Since $\rho\bar{u}=\rho\mathrm{U}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha}{dx}$ we have from equation (70)

$$\frac{d\rho\bar{u}}{dx}=0 \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

And since the action of the tube is symmetrical about the plane $x\,y$, we have at this plane

$$\frac{d\mathrm{U}}{dz}=0 \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Therefore, integrating between the limits z and 0, we have from equation (69)

$$\frac{dp}{dx}z=\frac{s}{\sqrt{\pi}}\rho\alpha\frac{d\mathrm{U}}{dz} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

Also, since s is constant across the tube, except within the layer over which the influence of the surface of the tube extends, and which is not taken into account, we have, integrating from z to c, and putting U_c for U at the surface

$$\tfrac{1}{2}\frac{dp}{dx}(c^2-z^2)=\frac{s}{\sqrt{\pi}}\rho\alpha(\mathrm{U}_c-\mathrm{U}) \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

From equation (43) we have, since $s\,\dfrac{d\rho\alpha}{dx}$ does not vary with z

$$\rho(\bar{u}-\bar{u}_c)=\rho(\mathrm{U}-\mathrm{U}_c) \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

Therefore, from equation (74)

$$u=-\frac{\sqrt{\pi}}{2s}\,\frac{1}{\rho\alpha}(c^2-z^2)\frac{dp}{dx}+\bar{u}_c \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

or putting

$$\Omega=\frac{\int_0^c \bar{u}\,dz}{\int_0^c dz} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

so that Ω is the mean velocity of the gas along the tube, we have integrating (76), and putting $p=\dfrac{\rho\alpha^2}{2}$

$$\frac{\Omega-\bar{u}_c}{\alpha}=\frac{\sqrt{\pi}}{6}\,\frac{c^2}{s}\,\frac{1}{p}\,\frac{dp}{dx} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

The relation between s and μ.

93. The only respect in which equation (78) differs from the usual equation between the motion of gas and the variation of pressure in a tube is that instead of μ we have

$$\frac{2}{\sqrt{\pi}}\,\frac{p}{\alpha}\,s.$$

For putting

$$\frac{dp}{dx}=\frac{d}{dz}\left(\mu\frac{d\bar{u}}{dz}\right)$$

we have for the usual equation

$$\Omega-\bar{u}_c=-\frac{c^2}{3\mu}\frac{dp}{dx} \quad \text{. . . . .} \quad (79)$$

and comparing (78) and (79)

$$\mu=\frac{2}{\sqrt{\pi}}\frac{p}{\alpha}s \quad \text{. . . . .} \quad (80)$$

The difference between equations (78) and (79) is, however, very important. For whereas μ is usually supposed to be constant, *i.e.*, independent of the diameter of the tube, it appears from (78) that such can only be the case so long as c is large as compared with s: s being a distance measured across the tube which by no variation in the condition of the gas can be made larger than the mean diameter of the tube.

This fact that s cannot increase beyond the diameter of the tube at once explains the anomalies (as they appeared to GRAHAM) between the times of transpiration for fine and coarse plugs.

The mean diameter of the interstices of GRAHAM's coarse plugs were so large, that with gas in the condition in which he used it, s was less than this diameter, and not being limited to the diameter of the tube was different for different gases and for different conditions of the same gas; whereas with the fine plugs, s being limited to the diameter of the tube, could no longer vary with the nature of the gas.

The limit to the value of s also indicates, what has been verified by the experiments described in Part I. of this paper, that the results which GRAHAM obtained with fine plates only, are to be obtained with coarse plates when the condition of the gas is such that s is limited by the diameter of the interstices.

The relation between s and the other properties of the gas.

94. The experiments made by GRAHAM and by MAXWELL in which the distances between the surfaces was such that there was no chance of s being limited by this distance, give consistent results, from which it has been found that

$$\mu \propto \frac{p}{\rho}.\text{*}$$

Hence taking $\mu=l\frac{p}{\rho}$ and substituting in equation (80) we have

$$s=\frac{\sqrt{\pi}}{2}\frac{\alpha}{\rho}l \quad \text{. . . . .} \quad (81)$$

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* Added Dec., 1879.—Subsequent observers have found that $\mu \propto \left(\frac{p}{\rho}\right)^{.77}$ so that MAXWELL's conclusions are not borne out.—See Phil. Trans., Part I., 1879, p. 240. This makes no difference to the subsequent part of this investigation, as no further use is made of equation (81).

From which it appears that in the same gas

$$s \propto \frac{a}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

when not limited by the solid objects.

The general case of transpiration.

95. The equation (78) is obtained on the assumption that s is so small compared with the diameter of the tube, that the layer of gas through which the influence of the surface of the tube extends may be neglected, and hence this equation cannot be taken as the law of transpiration when s_z comes to be limited by the diameter of the tube. And besides this, it is necessary to consider the value of $\overline{u_c}$, which cannot be done without considering the layer of gas throughout which the effect of discontinuity at the surface extends.

In order to take the discontinuity at the surfaces $z = \pm c$ into account, the values of $\sigma_x(\text{M})$ and $\sigma_z(\text{M}u)$ must be taken from equations (53) and (54a). These values substituted in equations (66) and (67) give equations which correspond to equations (69) and (70), but which involve the quantity $s_1 - s_2$, which quantity it will be well to examine before proceeding to the substitution.

The value of $s_1 - s_2$.

96. Remembering that s_1 and s_2 are taken respectively to represent the mean range of the quantity Q for the groups of molecules which have w respectively positive and negative, and taking $s'_1\ s'_2$ to represent the values of $s_1\ s_2$ at the surface $z = c$, we may express $s_1 - s_2$ as a function of s, c, and z.

The fact that $s_1 = s_2 = s$ when the point considered is without the range of the influence of the surface, shows that whatever may be the value of $s'_1 - s'_2$, $s_1 - s_2$ gradually diminishes as the point considered recedes from the solid surface, until at some distance depending on s at which the mean range is unaffected by the surface $s_1 - s_2 = 0$. It also appears from the fact of the gas being symmetrical about the axis of the tube that $s_1 - s_2$ is zero at the axis, so that even if the value of s_1 is limited by the surface, s_2 approximates to s_1 as the point considered approaches the axis of the tube.

The definite manner in which $s_1 - s_2$ varies across the tube could only be deduced by taking into account the distribution of velocities amongst the molecules; but as $s_1 - s_2$ must change after a continuous manner from one surface to another, we may take for an illustration, or even for an approximation, any law of variation which fits the extremes.

Such a law is given by

$$s_1 - s_2 = (s'_1 - s'_2)\frac{e^{-\frac{c-z}{a_1 s}} - e^{-\frac{c+z}{a_1 s}}}{1 - e^{-\frac{2c}{a_1 s}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

in which a_1 is a numerical factor depending only on the nature of the gas.

For the sake of distinctness it will for the present be assumed that s_1-s_2 has the values given by equation (83).

The velocity of the gas at the solid surface.

97. Putting q for the tangential force on the solid surfaces $z=\pm c$, we have

$$q=\sigma_z(\mathrm{M}u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

and by equations (53a) and (54a)

$$q=\frac{\rho\alpha}{2\sqrt{\pi}}(\bar{u}'_1-\bar{u}'_2)-\frac{s'_1-s'_2}{2\pi}\rho\alpha\frac{d\alpha}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

Also since $\frac{dp}{dx}$ is constant over the section, we have for the equilibrium of the fluid between two perpendicular sections of the tube at distance dx

$$q=-mc\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

where mc is the hydraulic mean depth of the tube (in the case of a flat tube $m=1$); therefore

$$\frac{\rho\alpha}{2\sqrt{\pi}}(\bar{u}'_1-\bar{u}'_2)=\frac{s'_1-s'_2}{2\pi}\rho\alpha\frac{d\alpha}{dx}-mc\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

Then if $\bar{u}_c$ is the velocity along the solid surface, we have

$$2\bar{u}_c=\bar{u}'_1+\bar{u}'_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

And since $\bar{u}'_2$ is the mean velocity after encounter at the surface of the tube, we may put

$$\bar{u}'_2=f\bar{u}'_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

where f is a factor depending on the nature of the impact at the surface. Hence

$$2u_c=\frac{1+f}{1-f}(\bar{u}'_1-\bar{u}'_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

or putting

$$\lambda=\frac{1+f}{1-f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90a)$$

$$2\bar{u}_c=\lambda(\bar{u}'_1-\bar{u}'_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90b)$$

And from equation (87)

$$\frac{\rho\alpha}{\sqrt{\pi}}\bar{u}_c=\lambda\left\{\frac{s'_1-s'_2}{2\pi}\rho\alpha\frac{d\alpha}{dx}-mc\frac{dp}{dx}\right\}. \qquad (91)$$

The coefficient of friction at the solid surface.

98. Since f or λ is important as regards that which is to follow, it is necessary to determine, as far as possible, on what these factors depend. I am not aware that any very definite idea has hitherto been arrived at as to the action between the molecules of a gas and a solid surface over which the gas may be in motion. It appears to have been thought sufficient in most cases to assume that the gas in immediate contact with the surface is at rest, which supposition is equivalent to neglecting any small motion there may be.

We see at once that the gas at the surface must have a velocity when the gas further away is in motion. For by our fundamental assumption the molecules which approach the surface will partake of the motion further away; so that even supposing the surface to be perfectly rough, the entire group, consisting of the approaching and receding molecules, would have a velocity equal to half that of the approaching molecules.

If the surface be less than perfectly rough, we have, as in equation (89),

$$\bar{u}'_2=f\bar{u}'_1$$

where f^{-1} may be considered to be the coefficient of roughness.

Since we have nothing in nature analogous to perfect roughness, we may assume that f is not zero, and the question arises whether f may not largely depend on the angle at which the molecules approach the plane.

Even if the solid surface were a perfectly even plane, $\frac{1}{f}$ would not be the simple coefficient of friction, but must also be a function of the force with which the molecules strike the surface, and the more nearly perpendicular to the surface was the direction of approach the smaller would be the value of f.

Whereas if, as seems highly probable, the action between the molecules and the surface is closely analogous to that between a ball and an uneven but perfectly smooth elastic surface, then for molecules approaching the surface at very small angles f would be unity, while for those approaching in a manner nearly perpendicular f would be zero, or nearly so.

The variation of f with the angle of approach can be of no particular moment so long as there is a sufficient thickness of gas between the surface considered and any surface which may be opposite, for in that case the mean angle of approach must be the same, whatever may be the condition of the gas. But when the gas is between two surfaces, as in a tube, and these surfaces are so near that the molecules range across the interval, then the fact that if small the angle of reflection (measured from the normal) will always be less than the angle of incidence, must cause the molecules to assume directions more and more nearly perpendicular to the surface as the tube becomes narrower, until some limit is reached.

The case of a billiard ball started obliquely along the table will serve to illustrate this. Each time the ball leaves the side cushions its path will be more nearly perpendicular, and if it could maintain its velocity, and the table was sufficiently long, it would eventually be moving directly across the table. This, however, would not be the final condition if the cushions were zigzag, for then a number of balls, in whatever direction they might be started, would finally attain a certain mean obliquity, depending on the unevenness of the cushions. And it would seem probable that the latter case must be that of the molecules in a tube so narrow that they can range across.

The ability of the molecules to range across the tube will depend on the value of $\frac{c}{s}$; hence it would appear that the most probable assumption with regard to the nature of λ is that

$$\lambda=\lambda_1 f_1\left(\frac{c}{s}\right)+\lambda_2 f_2\left(\frac{c}{s}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91a)$$

where $f_1\left(\frac{c}{s}\right)$ and $f_2\left(\frac{c}{s}\right)$ are functions of some such form as $e^{-\left(\frac{c}{s}\right)^2}$, $e^{-\left(\frac{s}{c}\right)^2}$ having respectively the values unity and zero when $\frac{c}{s}=0$, and zero and unity when $\frac{c}{s}=\infty$; and λ_1 is a coefficient independent of the nature of the gas on which λ_2 may depend.

That there is good reason for making this assumption appears from the comparison of the results for hydrogen and air (see result VIII., Art. 106).

The equations of motion as affected by discontinuity.

99. Substituting in equation (66) from equation (54a), and putting $\frac{dp}{dx}$ for $\frac{d}{dx}\sigma_x(\mathrm{M}u)$ as in Art. 92, we have for steady momentum along the tube

$$\frac{dp}{dx}+\frac{d}{dz}\left\{-\frac{s\rho\alpha}{\sqrt{\pi}}\frac{d\overline{u}}{dz}-\frac{s_1-s_2}{2\pi}\rho\alpha\frac{d\alpha}{dx}\right\}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Whence integrating between the limits 0 and z

$$z\frac{dp}{dx}=\frac{s\rho\alpha}{\sqrt{\pi}}\frac{d\overline{u}}{dz}+\frac{s_1-s_2}{2\pi}\rho\alpha\frac{d\alpha}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

And substituting for s_1-s_2 from equation (83)

$$\frac{s\rho\alpha}{\sqrt{\pi}}\frac{d\overline{u}}{dz}=z\frac{dp}{dx}-\frac{e^{-\frac{c-z}{\alpha_1 s}}-e^{-\frac{c+z}{\alpha_1 s}}}{1-e^{-\frac{2c}{\alpha_1 s}}}(s'_1-s'_2)\frac{\rho\alpha}{2\pi}\frac{d\alpha}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

Integrating equation (94) between the limits c and z we have

$$\frac{s\rho\alpha}{\sqrt{\pi}}(\bar{u}-\bar{u}_c)=-\frac{c^2-z^2}{2}\frac{dp}{dx}+a_1s\left\{\frac{1+e^{-\frac{2c}{a_1s}}}{1-e^{-\frac{2c}{a_1s}}}-\frac{e^{-\frac{c-z}{a_1s}}+e^{-\frac{c+z}{a_1s}}}{1-e^{-\frac{2c}{a_1s}}}\right\}(s'_1-s'_2)\frac{\rho\alpha}{2\pi}\frac{d\alpha}{dx} \quad . \quad (94a)$$

Integrating again between the limits 0 and c and putting $\Omega=\frac{\int_0^c\bar{u}dz}{c}$, we have, substituting for $\bar{u}_c$ from equation (91),

$$\frac{s\rho\alpha}{\sqrt{\pi}}\Omega=-\left(\frac{c^2}{3}+smc\lambda\right)\frac{dp}{dx}$$

$$+\left(a_1s\frac{1+e^{-\frac{2c}{a_1s}}}{1-e^{-\frac{2c}{a_1s}}}-\frac{a_1^2s^2}{c}-s\lambda\right)(s'_1-s'_2)\frac{\rho\alpha}{2\pi}\frac{d\alpha}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

100. Equation (95) is the equation of transpiration in a flat tube on the assumption that

$$s_1-s_2=\frac{e^{-\frac{c-z}{a_1s}}-e^{-\frac{c+z}{a_1s}}}{1-e^{-\frac{2c}{a_1s}}}(s'_1-s'_2)$$

A slight modification however is all that is necessary to render the equation perfectly general.

The only way in which the shape of the tube enters into the equation is in the coefficient of the first time on the right-hand side, *i.e.*, the coefficient of $\frac{dp}{dx}$, and whatever may be the shape of the tube this coefficient will be of the same form as far as the linear dimensions of the tube are involved, the only possible difference being in the numerical coefficients of c^2 and $sc\lambda$. Therefore if c^2 be multiplied by a coefficient A, which depends on the shape of the tube, since m also varies with the shape of the tube, we have for the general coefficient of $\frac{dp}{dx}$

$$sc\left(\mathrm{A}\frac{c}{s}+m\lambda\right)$$

As regards the coefficient of $\frac{d\alpha}{dx}$, this is affected by the assumption as to the particular form of (s_1-s_2); and if we assume a general form for s_1-s_2, such as

$$s_1-s_2=\Sigma\left\{\frac{e^{-\left(\frac{c-z}{a_1s}\right)^n}-e^{-\left(\frac{c+z}{a_1s}\right)^n}}{1-e^{-\left(\frac{2c}{a_1s}\right)^n}}\right\}(s'_1-s'_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

the coefficient of the last term would still be of the form

$$s\left\{f\left(\frac{c}{s}\right)+\lambda\right\}$$

wherein $f\left(\frac{c}{s}\right)$ varies continuously as $\frac{c}{s}$ varies from 0 to ∞, having a finite value when $\frac{c}{s}$ is infinite and being zero of the order $\frac{c}{s}$ when $\frac{c}{s}$ is zero.

101. The factor $s'_1-s'_2$ is clearly a function of c, $\frac{c}{s}$ and λ_3, where λ_3 depends on the nature of the impacts between the gas and the tube. And, moreover, when $\frac{c}{s}$ is small and the molecules cross the tube without encounter, $s'_1-s'_2$ is proportional to c—it may be shown that in the case of a flat tube $s_1-s_2=\pi mc$, and in the case of a round tube $s_1-s_2=\pi mc\left(1+\frac{2}{\pi}\right)$, for tubes of other shapes s_1-s_2 would have an intermediate value —so in this case we put

$$s'_1-s'_2=\pi m'c.$$

Again, where $\frac{c}{s}$ is large, then $s'_1-s'_2$ is equal to $s\lambda_3$.

Hence, as a perfectly general form for $s'_1-s'_2$, we have

$$s'_1-s'_2=\pi m'cf_3\left(\frac{c}{s}\right)+s\lambda_3f_4\left(\frac{c}{s}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

wherein $f_3\left(\frac{c}{s}\right)$ is zero when $\frac{c}{s}$ is large, and unity when $\frac{c}{s}$ is small; while $f_4\left(\frac{c}{s}\right)$ is unity when $\frac{c}{s}$ is large, and zero when $\frac{c}{s}$ is small.

The general equation of transpiration.

102. Substituting in equation (95) from equations (96) and (97) we have

$$\left.\begin{aligned}\rho\alpha\Omega=&-\sqrt{\pi}c\left\{A\frac{c}{s}+m\lambda\right\}\frac{dp}{dx}\\&+\frac{c}{2\sqrt{\pi}}\left\{\pi m'f_3\left(\frac{c}{s}\right)+\frac{s}{c}\lambda_3f_4\left(\frac{c}{s}\right)\right\}\left\{f\left(\frac{c}{s}\right)+\lambda\right\}\rho\alpha\frac{d\alpha}{dx}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (98)$$

Or since Art. 72 $p=\frac{\rho\alpha^2}{2}$, $\frac{\tau}{M}=K^2\alpha^2$, and Art. 98 $\lambda=\lambda_1f_1\left(\frac{c}{s}\right)+\lambda_2f_2\left(\frac{c}{s}\right)$ we have, remembering that M is constant,

$$\left.\begin{aligned}\frac{\Omega}{\sqrt{\frac{2p}{\rho}}}=&-\frac{\sqrt{\pi}}{2}c\left\{\mathrm{A}\frac{c}{s}+m\lambda_1 f_1\left(\frac{c}{s}\right)+m\lambda_2 f_2\left(\frac{c}{s}\right)\right\}\frac{1}{p}\frac{dp}{dx}\\&+\frac{c}{4\sqrt{\pi}}\left\{\pi m' f_3\left(\frac{c}{s}\right)+\frac{s}{c}\lambda_3 f_4\left(\frac{c}{s}\right)\right\}\left\{f\left(\frac{c}{s}\right)+\lambda_1 f_1\left(\frac{c}{s}\right)+\lambda_2 f_2\left(\frac{c}{s}\right)\right\}\frac{1}{\tau}\frac{d\tau}{dx}\end{aligned}\right\} \quad . \quad (99)$$

in which

A depends only on the shape of the tube and is $\frac{1}{3}$ for a flat tube,

$f\left(\frac{c}{s}\right)$ is of the order $\frac{c}{s}$ when $\frac{c}{s}$ is zero and is finite when $\frac{c}{s}$ is infinite,

$f_1\left(\frac{c}{s}\right)$ and $f_3\left(\frac{c}{s}\right)$ are unity when $\frac{c}{s}$ is zero and zero when $\frac{c}{s}$ is infinite, and

$f_2\left(\frac{c}{s}\right)$ and $f_4\left(\frac{c}{s}\right)$ are zero when $\frac{c}{s}$ is zero and unity when $\frac{c}{s}$ is infinite; all the functions varying continuously between the limits here ascribed.

Also λ_1 depends on the nature of the surface but not upon the nature of the gas, while λ_2 and λ_3 may depend both upon the gas and the surface.

Putting

$$\left.\begin{aligned}&\mathrm{F}_1\left(\frac{c}{s}\right)=\frac{\sqrt{\pi}}{2}\left[\mathrm{A}\left(\frac{c}{s}\right)+m\left\{\lambda_1 f_1\left(\frac{c}{s}\right)+\lambda_2 f_2\left(\frac{c}{s}\right)\right\}\right]\\ \text{and}\quad&\\&\mathrm{F}_2\left(\frac{c}{s}\right)=\frac{1}{4\sqrt{\pi}}\left\{\pi m' f_3\left(\frac{c}{s}\right)+\frac{s}{c}\lambda_3 f_4\left(\frac{c}{s}\right)\right\}\left\{f\left(\frac{c}{s}\right)+\lambda_1 f_1\left(\frac{c}{s}\right)+\lambda_2 f_2\left(\frac{c}{s}\right)\right\}\end{aligned}\right\} \quad . \quad . \quad (100)$$

we have for the general form of the equation of transpiration

$$\Omega=-c\sqrt{\frac{2p}{\rho}}\left\{\mathrm{F}_1\left(\frac{c}{s}\right)\frac{1}{p}\frac{dp}{dx}-\mathrm{F}_2\left(\frac{c}{s}\right)\frac{\mathrm{M}}{\tau}\frac{d\left(\frac{\tau}{\mathrm{M}}\right)}{dx}\right\} \quad . \quad . \quad . \quad . \quad . \quad (101)$$

SECTION X.—VERIFICATION OF THE GENERAL EQUATION OF TRANSPIRATION.

103. In this section the general equation obtained in Section IX. is applied to the particular cases of transpiration which have been the subject of experiments. It will thus appear how I was led to infer the results, and thence to make the experiments.

A summary of the experimental results has already been given in Art. 9, but for the immediate purposes the results may be stated as follows :—

Experimental results.

I. The law of corresponding results at corresponding densities, shown by the fitting of the logarithmic homologues. (See Arts. 28 and 40.)

II. The gradual manner in which the results varied as the density increased, shown by the continuous curvature of the curves which express these results. (See Plates 47, 48, and 49.)

III. The uniformity in direction in which both the *time of transpiration* under pressure (see Tables XIV. to XVII.), and the ratio of the *thermal differences of pressure to the mean pressure* (see Tables III. to XIII.) vary as the density increased.

IV. The fact, sufficiently proved by GRAHAM, that, *cœteris paribus*, the times of transpiration are proportional to the ratio of the differences of pressure to the mean pressure, the difference of pressure being small.

V. The fact, to a certain extent taken for granted, that the ratio of the *thermal differences* of pressure to the mean pressure are, *cœteris paribus*, proportional to the ratio of the difference of temperature to the absolute temperature, this ratio being small.

VI. The continual approximation towards constancy of the time of transpiration under pressure as the density diminished. (See Tables XIV. to XVII., and diagram 1, Plate 47.)

VII. The relation between the ultimate values of the times of transpiration for different gases (air and hydrogen) for small densities; the times are proportional to the square roots of their atomic weights. (See Art. 42.)

VIII. The fact that the times of transpiration for the same gas in capillary tubes, and at considerable densities, are inversely as the density and independent of the temperature.—MAXWELL* and GRAHAM.†

IX. The difference in the variation of the times of transpiration for different gases, shown by the fact that the logarithmic curves for hydrogen cannot be made to fit those for air. (Plates 47, 48, and 49.)

X. The approximation towards a constant value of the ratio which the *thermal differences of pressure* bear to the *mean pressure* as the density diminishes, whatever be the gas or plate, the ratio is that of the difference of the square roots of the absolute temperatures to the square root of the absolute temperature.

XI. The approximation, as the density increases, to a linear relation between the thermal differences of pressure and the reciprocal of the density.

XII. The difference between the law of variation of the thermal differences of pressure for different gases, as shown by the non-agreement of the logarithmic homologues for air and hydrogen. (Plates 48 and 49.)

XIII. The transpiration of a varying mixture of gases through a porous plate. —Investigated by GRAHAM.

104. In order to bring out the agreement of the experimental results with those deduced from the equation, we put

$$\frac{\nabla}{\Omega} \text{ for the time of transpiration.}$$

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* Phil. Trans. 1866, pp. 249–268, also note to Art. 94.

† Phil. Trans., 1849, pp. 349–362.

$b\frac{dp}{dx}$ for the difference of pressure on the two sides of the plate.

$b\frac{d\tau}{dx}$ for the difference of temperature on the two sides of the plate.

The suffix s will be used to distinguish quantities relating to the stucco plate, and m to distinguish those relating to meerschaum.

x y are the coordinates of a point on any one of the curves on fig. 8, Art. 40, or Plate 48, which are the logarithmic homologues of the experimental curves.

105. The experimental result I. follows from the general form of equation (101). For putting, as in the experiment on transpiration under pressure, $\frac{d\tau}{dx}=0$ and M and $\frac{1}{p}\frac{dp}{dx}$ constant, equation (101) becomes

$$\Omega=-c\,\mathrm{F}_1\left(\frac{c}{s}\right)\sqrt{\frac{\kappa^2\tau}{\mathrm{M}}}\,\frac{1}{p}\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

The times of transpiration are proportional to $\frac{1}{\Omega}$ for the same tube or plate, and if ∇ be a factor depending on the number and size of the openings through the plate, we have the time of transpiration equal to $\frac{\nabla}{\Omega}$.

Putting

$$y=\log\frac{\nabla}{\Omega},\quad x=\log\frac{1}{s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

And indicating the quantities referring to particular plates by s and m, we have

$$\left.\begin{aligned} x_s+\log c_s&=\log\frac{c_s}{s_s}\\ x_m+\log c_m&=\log\frac{c_m}{s_m}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

Whence taking the coefficients A, m, λ_1, λ_2, to be the same for stucco as for meerschaum (see Appendix, note 4), it follows from equation (102) that when $\frac{c_s}{s_s}=\frac{c_m}{s_m}$

$$\text{and}\qquad \left.\begin{aligned} x_s&=x_m+\log\frac{c_m}{c_s}\\ y_s&=y_m-\log\frac{c_s\nabla_m}{c_m\nabla_s}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

Hence we see that the curves expressing the relation between the logarithms of the reciprocals of the mean ranges and the logarithms of the times of transpiration must have the same shape for different plates, such as stucco and meerschaum. And, moreover, that the difference between the abscissæ of corresponding points for the different plates is the logarithm of the ratio of the coarseness of the plates whatever may be the nature of the gas.

In the experiments we have an exactly similar agreement between the curves expressing the log. (densities) and the log. (times).

Hence the only point of difference between the results deduced from the equation and those derived from the experiments is, that the one depends on $\frac{1}{s}$ and the other upon ρ—the temperature being constant. Whereas it appears not only as in Art. 93, but in whichever way we examine s, that however s may vary with the molecular mass and with the temperature, it must be inversely proportional to the density.

Therefore the fitting of the logarithmic curves is a direct inference from the form of the general equation (101).

We also see that the common difference in the abscissæ of the curves deduced from the equation is the logarithm of the ratio of the diameters of the interstices; and hence we infer that the difference in the abscissæ of the experimental curves for meerschaum and stucco gives the ratio of the mean diameters of the interstices in these plates. (See Appendix, note 4.)

The common difference in the ordinates is, according to the equation, the logarithm of the ratio $\frac{\nabla_m c_s}{c_m \nabla_s}$; and although ∇_m and ∇_s are unknown, the experiments verify the theory in as much as they show that the common difference is independent of the nature of the gas—the same difference being obtained with hydrogen as with air—and depends entirely on the plates.

The fitting of the curves which express the logarithms of the thermal differences of pressure follows in a precisely similar manner from equation (101).

In these experiments $\Omega=0$ and $\frac{1}{\tau}\frac{d\tau}{dx}$ and M were constant, so that equation (101) becomes

$$F_1\left(\frac{c}{s}\right)\frac{1}{p}\frac{dp}{dx}=F_2\left(\frac{c}{s}\right)\frac{M}{\tau}\frac{d\tau}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

And putting

$$y=\log\frac{dp}{dx}$$

$$x=\log\frac{1}{s}$$

we have as in the previous case, supposing the coefficient in F_1 and F_2 to be the same for stucco as for meerschaum (see Appendix, note 4), where $\frac{c_s}{s_s}=\frac{c_m}{s_m}$

$$\left.\begin{aligned} x_s&=x_m+\log\frac{c_m}{s_s} \\ y_s&=y_m+\log\frac{p_m}{p_s} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

And since τ and M are the same for both plates

$$\frac{p_m}{p_s}=\frac{\rho_m}{\rho_s}.$$

Hence in this case, according to the general equation (106), the common difference in the ordinates of corresponding points is the logarithm of the ratios of corresponding densities, while the difference in the abscissæ is the logarithm of the ratio of the coarseness of the plates which is the reciprocal of the ratios of the mean ranges. If, therefore, as has just been assumed, the densities are proportional to the mean ranges, the common difference of the ordinates should be the same as that of the abscissæ, and the same for these curves as for those of transpiration under pressure.

Thus we have excellent opportunities of verifying the conclusion that s varies inversely as ρ, and the indication as to the manner in which c enters into the relation between dp and $d\tau$.

This verification is complete, for although there is a slight discrepancy between the common difference for the ordinates and that for the abscissæ, this, as has been explained in Art. 30, was in all probability owing to certain discrepancies in the difference of temperature maintained on the two sides of the plates (see Appendix, note 4). And even if unexplained these discrepancies are small enough to be neglected.

The actual differences are as follows:—

	Thermal Transpiration.		Transpiration.
Plates.	Abscissæ.	Ordinates.	Abscissæ.
Meerschaum No. 3, and Stucco No. 1	·698	·775	
„ „ „ „ 2	·745	·890	·819

Thus the dependence of transpiration on the ratio $\frac{c}{s}$ first revealed by the theory as expressed in equation (101) has been completely verified by the experiments of transpiration under pressure, and on thermal transpiration. And it must be noticed that while the verification has been obtained both for hydrogen and air, the experiments on either gas suffice for complete verification. And thus the exact agreement of the common differences both of ordinates and abscissæ for the two gases (although the absolute ordinates differ widely, and the shapes of the curves differ considerably) not only affords a double verification, but precludes the possibility of accidental coincidence.

It is further to be noticed, both with respect to the foregoing comparison of the theoretical with the experimental results, and also with respect of such further comparisons as will be made, that the reasoning admits of being reversed; and instead of deducing the experimental results from the equation, it might have been shown that a similar equation is the necessary outcome of the experimental results. Indeed, this has been already done, and it is only out of regard to the length of this paper that I refrain from including the inverse reasoning.

106. The experimental results II. and III. follow at once from the fact that the

various functions of $\frac{c}{s}$ in equation (101) increase or decrease continuously between the values $\frac{c}{s}=0$ and $\frac{c}{s}=\infty$.

Results IV. and V. also follow so directly from equation (101) as to require no comment.

Results VI. and VII. refer to transpiration under pressure when $\frac{c}{s}$ is small. Under these circumstances, since $\frac{d\tau}{dx}=0$, equation (99) becomes

$$\frac{\Omega}{\sqrt{\frac{\kappa^2\tau}{\mathrm{M}}}}=-\frac{\sqrt{\pi}}{2}cm\lambda_1\frac{1}{p}\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

and taking, as in the experiments, τ and $\frac{1}{p}\frac{dp}{dx}$ constant, we have for the same plate

$$\Omega \propto \frac{m\lambda_1}{\sqrt{\mathrm{M}}},$$

which is result VI.

And assuming, as in Art. 98, that $m\lambda_1$ is independent of M or any property of the gas, we have

$$\Omega \propto \frac{1}{\sqrt{\mathrm{M}}},$$

and therefore the times of transpiration of the different gases through the same plate are proportional to the square roots of the molecular weights, which is experimental result VII.

This result, therefore, verifies the conclusion arrived at in Art. 98, that when the tube is small compared with s the effect of the impacts at the surface is independent of the nature of the gas.

Result VIII. relates to transpiration under pressure when $\frac{c}{s}$ is large.

Then we have from equation (99)

$$\frac{\Omega}{\sqrt{\frac{\kappa^2\tau}{\mathrm{M}}}}=-\frac{\sqrt{\pi}}{2}c\left(\mathrm{A}\frac{c}{s}+m\lambda_2\right)\frac{1}{p}\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

Therefore since τ, M, and $\frac{1}{p}\frac{dp}{dx}$ are to be taken as constant; when $\frac{c}{s}$ becomes sufficiently large

$$\Omega \propto \frac{1}{s},$$

that is

$$\Omega \propto \rho\,;$$

and this is result VIII.

In order to compare different gases we have, when $\frac{c}{s}$ is sufficiently large,

$$\Omega = -\frac{\sqrt{\pi}}{2}\sqrt{\frac{\kappa^2\tau}{\mathrm{M}}}\,\mathrm{A}\,\frac{c^2}{s}\,\frac{1}{p}\,\frac{dp}{dx} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (110)$$

Therefore

$$\Omega \propto \frac{1}{s\sqrt{\mathrm{M}}}.$$

This gives the relative values of s for different gases; as, for instance, air and hydrogen. GRAHAM found that the times of transpiration of these gases through a capillary tube are in the ratio 2·04. The ratio of the square roots of the molecular weights is 3·8. Hence at equal pressures and equal temperatures the mean range for hydrogen is to the mean range for air as 3·8 is to 2·04.

It appears, however, at once from the equation that these ratios are not constant unless $\frac{c}{s}$ is very large. As $\frac{c}{s}$ diminishes, the term involving λ_2 becomes important, and it is to this term we must look for the explanation of the result IX.—the marked non-correspondence of the curves for hydrogen and air. If λ_2 depends on the nature of the gas then this difference in shape is accounted for, which confirms the conclusion of Art. 98 that when the tube is large compared with s the effect of the impacts at the surface will probably depend on the nature of the gas.

107. Result X. refers to the thermal differences of pressure when $\frac{c}{s}$ is small.

In this case $\Omega = 0$, while $\frac{1}{\tau}$, $\frac{d\tau}{dx}$, and M are constant.

Equation (99) becomes

$$\frac{1}{p}\frac{dp}{dx} = \frac{1}{2}\frac{m'}{m}\frac{1}{\tau}\frac{d\tau}{dm} = \frac{m'}{m}\frac{1}{\sqrt{\tau}}\frac{d\sqrt{\tau}}{dx} \quad . \; . \; . \; . \; . \; . \; . \quad (111)$$

The exact relation between m and m' would appear, as explained in Art. 101, to depend on the shape of the section of the tube, and to be somewhere between 1 and $1+\frac{2}{\pi}$, its respective values for a flat and round tube. This view, however, is based on the assumption that the molecules are uniformly distributed as regards direction, whereas it appears probable, from reasoning similar to that of Art. 98, that the molecules tend to assume a direction normal to the surface, and in this case for a tube of curvilinear section the value of $\frac{m'}{m}$ would be reduced.

According to the experiments, it appears that as the density diminishes, $\frac{m'}{m}$ approaches to unity; but owing to the impossibility of measuring the exact difference on the two sides of the plate this determination is not very definite.

Result XI. refers to the thermal difference of pressure when $\frac{c}{s}$ is large.

In this case $\Omega=0$, while $\frac{1}{\tau}$, $\frac{d\tau}{dx}$, and M are constant.

Equation (99) becomes

$$\left(\mathrm{A}\frac{c}{s}+m\lambda_2\right)\frac{1}{p}\frac{dp}{dx}=\frac{1}{2\pi}\frac{s}{c}\left(f\left(\frac{c}{s}\right)+\lambda_2\right)\frac{1}{\tau}\frac{d\tau}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

in which $f\left(\frac{c}{s}\right)$ has some finite value.

In the limit, therefore, we may neglect $m\lambda_2$, and we have

$$\mathrm{A}\frac{1}{p}\frac{dp}{dx}=\frac{1}{2\pi}\frac{s^2}{c^2}\left(f\left(\frac{c}{s}\right)+\lambda_2\right)\frac{1}{\tau}\frac{d\tau}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

And since $s\propto\frac{1}{\rho}$ and c is constant

$$\frac{1}{p}\frac{dp}{dx}\propto\frac{1}{\rho^2}\frac{1}{\tau}\frac{d\tau}{dx}$$

which is result XI.

Since the coefficient of $\frac{1}{\tau}\frac{d\tau}{dx}$ in equation (113) involves λ_2, which (Art. 98) depends on the nature of the gas, this equation indicates that different results would be obtained with different gases.

And this appears still more in the case of intermediate pressures when $m\lambda_2$ on the left of equation (112) is important.

These conclusions are according to result XII., which therefore affords additional proof of the correctness of the conclusions in Art. 98 respecting the value of λ_2.

108. I have now shown how I was led to predict the experimental results, and how in every particular the experiments have verified the theory, both as regards transpiration under pressure and the thermal differences of pressure. This concludes the application of the theory to those experimental results of transpiration which were revealed by the theory.

There remains, however, an important class of transpiration phenomena of which, as yet, no mention has been made. These are the phenomena of transpiration when the gas on the two sides of the plate differs in molecular constitution.

Transpiration by a variation in the molecular condition of the gas.

108A. These phenomena are well known, and were experimentally investigated by GRAHAM, but hitherto, I believe, no complete theoretical explanation of them has been given. The diffusion of one gas into another has been explained by MAXWELL; but what has not, so far as I know, been explained is, that there should result a current from the side of the denser to that of the lighter gas. Indeed, from the manner in which these phenomena have been for the most part described, it would appear that the importance of this current has been overlooked; for, owing to the fact that a larger volume of the lighter gas passes, the phenomena are generally described as if the current were from the lighter to the denser gas.

These phenomena of transpiration, like those already considered, may be shown to follow directly from the theory. But as has been already mentioned in Art. 73, in order to completely adapt the equations of transpiration to the case of two or more gases, it would be necessary to commence by considering the case of two or more systems of molecules having different molecular weights, after the manner adopted by MAXWELL.* Such an adaptation of the equations is too long to be included in this paper; but it may be seen from the equations, as they have already been deduced, that these particular phenomena would, and in some cases do, follow.

Suppose that the gas on the two sides of the plate is at the same pressure and temperature, but that there is a difference in molecular constitution as air and hydrogen. Thus when the condition has become steady there will be a gradual variation of the molecular condition of the gas through the plate; in this case τ is constant and p is constant, but the mean value of M varies.

If we take M_1 and M_2 (as the molecular masses of the two systems of molecules), and consider a case in which M_1 differs but very slightly from M_2, equation (98) becomes

$$\rho\Omega = -c\mathrm{F}_2\left(\frac{c}{s}\right)\kappa\sqrt{\tau}\frac{\rho}{(\overline{\mathrm{M}})^{\frac{3}{2}}}\frac{d\overline{\mathrm{M}}}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

where $\overline{\mathrm{M}}$ is the mean mass of the molecules, or if ρ_1 and ρ_2 are the densities of the two gases

$$\overline{\mathrm{M}} = \frac{\mathrm{M}_1\mathrm{M}_2}{\mathrm{M}_2\rho_1 + \mathrm{M}_1\rho_2}(\rho_1 + \rho_2).$$

Whence, putting $\mathrm{N}_1 = \frac{\rho_1}{\mathrm{M}_1}$, $\mathrm{N}_2 = \frac{\rho_2}{\mathrm{M}_2}$,

$$\overline{\mathrm{M}} = \frac{\mathrm{N}_1\mathrm{M}_1 + \mathrm{N}_2\mathrm{M}_2}{\mathrm{N}_1 + \mathrm{N}_2}.$$

And since the pressure and temperature are constant

$$\mathrm{N}_1 + \mathrm{N}_2 = \mathrm{N}$$

where N is constant throughout the gas.

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* Phil. Trans., 1867.

Therefore

$$\frac{d\bar{M}}{dx}=\frac{M_1-M_2}{N}dN_1=\frac{d\bar{\rho}}{N};$$

and (114) becomes

$$\rho\Omega=-cF_2\left(\frac{c}{s}\right)\kappa\sqrt{\tau}\frac{1}{\sqrt{M}}\frac{d\rho}{dx} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (115)$$

If $\frac{c}{s}$ is small, then $F_2\left(\frac{c}{s}\right)=\frac{1}{4}m'\lambda_1$.

Hence in this case

$$\rho\Omega=-\frac{c}{4}m'\lambda_1\kappa\sqrt{\tau}\frac{1}{\sqrt{M}}\frac{d\rho}{dx} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (116)$$

And this is in exact accordance with GRAHAM's law, which is that the rate of transpiration is proportional to the difference in the square roots of the densities of the gas. For—

$$\frac{d\rho}{dx}=(M_1-M_2)\frac{dN_1}{dx},$$

and since M_1-M_2 is small

$$\frac{d\rho}{dx}=\sqrt{M}\{\sqrt{M_1}-\sqrt{M_2}\}\frac{dN_1}{dx},$$

or

$$\rho\Omega=-\frac{c}{2}m'\lambda_{,}\kappa\sqrt{\tau}\{\sqrt{M_1}-\sqrt{M_2}\}\frac{dN_1}{dx} \quad . \ . \ . \ . \ . \ . \quad (117)$$

This form of equation is obtained by neglecting the difference of M_1 and M_2; but by taking into account the two systems of molecules throughout the investigation, an equation similar to (117) would have been obtained without any such assumption.

Thus we see that the general equation of transpiration may be made to include not only the cases of transpiration under pressure and thermal transpiration, but also the well known phenomena of transpiration caused by the difference in the molecular constitution of the gas. And in this case, as in that of transpiration under pressure, the equation reveals laws connecting the results obtained with plates of different coarseness and different densities of gas, which doubtless admit of experimental verification.

This completes the explanation of the phenomena of transpiration through porous plates.

SECTION XI.—*Application to apertures in thin plates and impulsion.*

Condition of the gas.

109. When the gas within a vessel is in uniform condition, excepting in so far as it is disturbed by a steady flow of gas or of heat from what, compared with the size of

the vessel, may be considered as a small space, such as a small aperture in the side of the vessel or a small hot body within the vessel, the effect of such steady flow will be to cause a varying condition throughout the gas. The lines of flow, whether of heat or of gas, will diverge from the exceptional space, and the surfaces of equal pressure and temperature will be everywhere perpendicular to the lines of flow of matter and heat respectively. Except in the case of absolute symmetry, the lines of flow will not be straight, nor will the directions of the lines of flow in the immediate region of any point focus in a point.

But in the immediate neighbourhood of any point P, the direction of the lines of flow must be such that the directions of the lines of flow parallel to some plane, $x\,y$, will converge to some point C, while the directions of the lines of flow parallel to the perpendicular plane $x\,z$ will converge to some point C′ in CP, which it will be seen is taken parallel to the axis of x.

Whence putting $\mathrm{PC}=r_y$ and $\mathrm{PC}'=r_z$, the surface of equal pressure or temperature at P will be a surface perpendicular to x, and having r_y and r_z as its principal radii of curvature in the planes of $x\,y$ and $x\,z$ respectively. The simplest cases are those in which either the two radii are equal or one is infinite, and these are the cases which will for the most part be considered.

It will at once be seen that at any point within gas in the condition just described, ρ, α, $\overline{u^2}+\overline{v^2}+\overline{w^2}$ and $\mathrm{U}^2+\mathrm{V}^2+\mathrm{W}^2$ are functions of $r_y\,r_z$.

Also, remembering that the axis of x is taken in the direction of the lines of flow at P, the point considered, we see that at P, V, W, $\bar{v}$, $\bar{w}$, are severally zero, as are also $\frac{d\mathrm{U}}{dy}$, $\frac{d\mathrm{U}}{dz}$, $\frac{d^2\mathrm{V}}{dy^2}$, $\frac{d^2\mathrm{W}}{dz^2}$, $\frac{d\mathrm{V}}{dx}$, $\frac{d\mathrm{W}}{dx}$; while we have

$$\left.\begin{aligned}
\frac{d^2\mathrm{U}}{dy^2}&=\frac{1}{r_y}\frac{d\mathrm{U}}{dx}-\frac{1}{r_y^2}\mathrm{U},\\
\frac{d^2\mathrm{U}}{dz^2}&=\frac{1}{r_z}\frac{d\mathrm{U}}{dx}-\frac{1}{r_z^2}\mathrm{U},\\
\frac{d\mathrm{V}}{dy}&=\frac{\mathrm{U}}{r_y},\\
\frac{d\mathrm{W}}{dz}&=\frac{\mathrm{U}}{r_z},\\
\frac{d^2\mathrm{V}}{dxdy}&=\frac{1}{r_y}\frac{d\mathrm{U}}{dx}-\frac{1}{r_y^2}\mathrm{U},\\
\frac{d^2\mathrm{W}}{dxdz}&=\frac{1}{r_z}\frac{d\mathrm{U}}{dx}-\frac{1}{r_z^2}\mathrm{U}.
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(118)$$

Also putting $f(\rho\alpha)$ for any function of ρ and α, $\frac{d}{dy}f(\rho\alpha)$ and $\frac{d}{dz}f(\rho\alpha)$ are zero, while

$$\left.\begin{aligned}\frac{d^2f(\rho\alpha)}{dy^2}&=\frac{1}{r_y}\frac{df(\rho\alpha)}{dx},\\ \frac{d^2f(\rho\alpha)}{dz^2}&=\frac{1}{r_z}\frac{df(\rho\alpha)}{dx}.\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (119)$$

The equations of steady condition.

110. Equations (118) and (119), together with equations (43) to (47), enable us to obtain from equation (57) the equations of steady condition.

For steady density

$$\frac{d}{dx}\left\{r_yr_z\left(\rho\mathrm{U}-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha}{dx}\right)\right\}=0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (120)$$

For steady momentum putting, as before, $\frac{dp}{dx}=\frac{d}{dx}(\sigma_x(\mathrm{M}u))$

$$\frac{dp}{dx}-\frac{2s}{\sqrt{\pi}}\frac{d}{dx}\left\{\rho\alpha\mathrm{U}\left(\frac{1}{r_y}+\frac{1}{r_z}\right)\right\}=0 \; . \; . \; . \; . \; . \; . \; . \; . \quad (121)$$

For steady pressure

$$\frac{d}{dx}\left\{r_yr_z\left(\frac{5}{2}\rho\alpha^2\mathrm{U}-\frac{3}{\sqrt{\pi}}s\frac{d\rho\alpha^3}{dx}\right)\right\}=0 \; . \; . \; . \; . \; . \; . \; . \; . \quad (122)$$

These equations (120), (121), (122), might be treated in a manner similar to that in which the corresponding equations for the case of the tube were treated in Section IX., but for various reasons another method commends itself.

In the first place we cannot in this case ignore the condition of steady pressure, for there can be no lateral adjustment of temperature as in the case of the tube. (See Art. 91.) The physical meaning of this is, that in this case the condition of the gas cannot be supposed to vary uniformly even along the lines of flow. It must vary after a fixed law, and this fact restricts the conditions under which the equations can be considered to hold to points where $\frac{s}{r}$ is so small that $\left(\frac{s}{r}\right)^2$ may be neglected. So that any general result obtained from these equations would only apply to points at considerable distances from the foci C and C′.

Again, these equations as they stand include the case in which the flow of the gas may be caused by a considerable difference of pressure, as, for example, transpiration through a small aperture under pressure, whereas if we exclude this case we may, by neglecting such terms as $\frac{s^2}{\alpha^2}\overline{\frac{d\alpha}{dx}}\Big|^2$, very greatly simplify the equations without affecting their application to the cases which it is our principal object to explain.

These two cases are as follow—

1. The flow of gas through a small orifice in a thin plate when the mean pressure of the gas is the same on both sides of the plate, the flow being caused by a difference in temperature on the two sides of the plate, or a difference in the molecular condition of the gas.

2. The excess of pressure which the gas exerts on a small body when the body has a higher temperature than the gas.

Thermal transpiration through an aperture in a thin plate.

111. In this case, since there is no tangential stress, we have (Art. 87)

$$U=0.$$

Whence by equation (121)

$$\frac{dp}{dx}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122a)$$

Since $p=\rho\frac{\alpha^2}{2}$ we have, integrating equations (120) and (122), respectively

$$\left.\begin{aligned} r_y r_z \frac{d\alpha}{dx} &= \frac{\sqrt{\pi}}{2sp}\alpha^2 G, \\ r_y r_z \frac{d\alpha}{dx} &= -\frac{\sqrt{\pi}}{6sp} H. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122b)$$

G and H are constants, such that $\frac{\beta H}{2r_y r_z}$ is the rate at which heat is carried across a unit of area, and $\frac{G}{r_y r_z}$ is the rate at which matter is carried across.

From equation (122b) we have

$$H=-3\alpha^2 G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123)$$

Equation (123) can only be approximately true as α^2 is not constant; therefore the condition $U=0$ is not possible, *i.e.*, it is only approximately fulfilled, whence it follows that p is only approximately constant. The closeness of these approximations will depend on the variation of α^2 and within the limits of our approximation we may consider the condition to hold.

From equation (123) we see that the direction of flow of gas is opposite to that of the flow of heat, while since $\alpha^2 \propto \frac{\tau}{M}$, the rate of flow of gas is proportional to the flow of heat, to the mass of a molecule and inversely proportional to τ, the absolute temperature.

By equation (48) we have

$$\rho\bar{u}=-\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha}{dx},$$

or since $\rho\alpha^2$ is constant

$$\bar{u}=\frac{s}{\sqrt{\pi}}\frac{d\alpha}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

which it may be noticed is of the same form as results from the equation of transpiration through a tube when p is constant.

Thermal Impulsion.

112. In this case there is no motion, therefore

$$\bar{u}=0,$$

whence from equations (48)

$$\rho\mathrm{U}=\frac{s}{\sqrt{\pi}}\frac{d\rho\alpha}{dx} \quad . \quad . \quad . \quad . \qquad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

This satisfies equation (120).

Substituting from equation (125) in equations (121) and (122), and remembering that $p_x=\frac{\rho\alpha^2}{2}-\frac{2s}{\sqrt{\pi}}\frac{d}{dx}(\rho\alpha\mathrm{U})$, Art. 82, we find that these equations lead to the same result if $\rho\alpha^2$ is constant.

Putting $p_1=\frac{\rho\alpha^2}{2}$ we have from equation (122)

$$\frac{d}{dx}\left(r_y r_z s p_1\frac{d\alpha}{dx}\right)=0,$$

whence integrating

$$\frac{d\alpha}{dx}=-\frac{\sqrt{\pi}\mathrm{H}}{11r_y r_z}\frac{1}{sp_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

where H has the same value as in Art. 111.

Remembering that $\frac{dr_y}{dx}=\frac{dr_z}{dx}=1$, also considering s constant, we have differentiating

$$\frac{d^2\alpha}{dx^2}=\frac{\sqrt{\pi}}{11}\frac{\mathrm{H}}{sp_1}\frac{1}{r_y r_z}\left(\frac{1}{r_y}+\frac{1}{r_2}\right)$$

$$=-\frac{r_y+r_z}{r_y r_z}\frac{d\alpha}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

Also putting $r_y=r_z$, $\alpha=\alpha'$ when $r=\infty$ and $\alpha=\alpha_c$ when $r=c$, and integrating equation (126), we have

$$\left.\begin{aligned} \alpha-\alpha' &= \frac{\sqrt{\pi}}{11}\frac{\mathrm{H}}{sp_1}\frac{1}{r}, \\ &= \frac{c}{r}\cdot(\alpha_c-\alpha'), \\ &= \tfrac{1}{2}r^2\frac{d^2\alpha}{dx^2}. \end{aligned}\right\} \quad \dots\dots\dots \quad (128)$$

In a similar way we obtain from equation (121)

$$\frac{dp_x}{dx} = -\frac{4s^2}{\pi}p_1\frac{d}{dx}\left\{\left(\frac{1}{r_y}+\frac{1}{r_z}\right)\frac{1}{\alpha}\frac{d\alpha}{dx}\right\} \quad \dots\dots\dots \quad (129)$$

whence integrating

$$\frac{p_x-p_0}{p_1} = -\frac{4s^2}{\pi}\left\{\frac{1}{r_y}+\frac{1}{r_z}\right\}\frac{1}{\alpha}\frac{d\alpha}{dx}-\mathrm{C}_2 \quad \dots\dots\dots \quad (130)$$

and from (127)

$$\frac{p_x-p_0}{p_1} = \frac{4s^2}{\pi}\frac{1}{\alpha}\frac{d^2\alpha}{dx^2}-\mathrm{C}_2.$$

If r be infinite $p_x=p_0=p_1$ and $\frac{d^2\alpha}{dx^2}=0$. Therefore $\mathrm{C}_2=0$ and

$$\frac{p_x-p_0}{p_1} = \frac{4}{\pi}s^2\frac{1}{\alpha}\frac{d^2\alpha}{dx^2} \quad \dots\dots\dots \quad (131)$$

which result may be obtained directly from the value of p_x, Art. 82.

From equation (128)

$$\left.\begin{aligned} \frac{p_x-p_0}{p_1} &= \frac{8}{\pi}\frac{s^2}{r^2}\frac{\alpha-\alpha'}{\alpha} \\ &= \frac{8cs^2}{\pi r^3}\frac{\alpha_c-\alpha'}{\alpha} \end{aligned}\right\} \quad \dots\dots\dots \quad (132)$$

Putting $\frac{\tau}{\mathrm{M}}=\kappa^2\alpha^2$ and neglecting $\left.\frac{1}{\alpha}\overline{\frac{d\alpha}{dx}}\right|^2$ as compared with $\frac{d^2\alpha}{dx^2}$

$$\frac{p_x-p_1}{p_1} = \frac{2}{\pi}s^2\frac{\mathrm{M}}{\tau}\frac{d^2}{dx^2}\left(\frac{\tau}{\mathrm{M}}\right)^* \quad \dots\dots\dots \quad (133)$$

$$\left.\begin{aligned} &= \frac{8}{\pi}\frac{s^2}{r^2}\frac{\sqrt{\frac{\tau}{\mathrm{M}}}-\sqrt{\frac{\tau'}{\mathrm{M}}}}{\sqrt{\frac{\tau'}{\mathrm{M}}}} \\ &= \frac{8}{\pi}\frac{cs^2}{r^2}\frac{\sqrt{\frac{\tau_c}{\mathrm{M}}}-\sqrt{\frac{\tau'}{\mathrm{M}}}}{\sqrt{\frac{\tau'}{\mathrm{M}}}} \end{aligned}\right\} \quad \dots\dots\dots \quad (134)$$

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* From an abstract of a paper read before the Royal Society by Professor MAXWELL, in April, 1878 (see 'Nature,' May 9, 1878), I see that Professor MAXWELL has obtained an expression for this inequality

From equation (127) we have

$$\frac{p_x-p_1}{p_1}=\frac{8}{11}\frac{1}{\sqrt{\pi}}\frac{\mathrm{H}s}{p\alpha}\frac{r_y+r_z}{r_y^2r_z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

where, as before, $\frac{\beta\mathrm{H}}{2r_yr_z}$ is the quantity of heat carried across a unit of surface.

At points near to the surface.

113. In equations (131), (132), and (135) no account has been taken of the discontinuity in the immediate neighbourhood of the surface; hence the results obtained from these equations may not hold good within the layer of gas of thickness s, which is adjacent to the surface.

In order to take this discontinuity into account, the equations of steady conditions should be modified in the manner described in Art. 84, but for this particular case the same thing may be accomplished in a somewhat simpler manner.

Suppose the solid surface to be either spherical or cylindrical at the point considered, and put c_1 for the radius. Then it is obvious that when $\frac{c_1}{s}$ is very large the pressure on the surface will be but slightly affected by the layer immediately adjacent to the surface, *i.e.*, putting p_{c_1} for the pressure at the surface, and p_{c_1+s} for the pressure at a distance s from the surface, $1-\frac{p_{c_1}-p_1}{p_{c_1+s}-p_1}$ is small when $\frac{c_1}{s}$ is large.

When, however, the gas surrounding the surface is limited by another surface,

---

of pressure or "stress" arising from the inequality of temperature. The result given by Professor MAXWELL is

$$p_x-p_1=\frac{3\mu^2}{\rho\theta}\frac{d^2\theta}{dx^2}$$

where μ is the coefficient of viscosity, θ the absolute temperature, and x any one of the three directions x, y, z. This result, when transformed to the present notation, becomes

$$p_x-p_1=\frac{3\mu^2}{\rho\tau}\frac{d^2\tau}{dx^2}$$

And if we put, as in equation (80),

$$\mu=\frac{2}{\sqrt{\pi}}\frac{p_1s}{\alpha}$$

we have

$$\frac{p_x-p_1}{p_1}=\frac{6}{\pi}s^2\frac{1}{\tau}\frac{d^2\tau}{dx^2}$$

It is thus seen that the two results are identical in form, but that Professor MAXWELL makes the pressure just three times as great as that given by equation (133).

In the abstract published in 'Nature,' MAXWELL has not given the details of the method by which he arrived at his result.

(which for simplicity may be taken concentric and of radius c_2), then in order that $1-\frac{p_{c_1}-p_{c_2}}{p_{c_1+s}-p_{c_2-s}}$ may be small, we must have $\frac{c_1-c_2}{s}$ large as well as $\frac{c_1}{s}$

Our equations, therefore, may be seen to hold good when the radius of the solid surface is large compared with s and the distance between the opposite surfaces is also large.

On the other hand, in the limit when either $\frac{c_1}{s}$ or $\frac{c_1-c_2}{s}$ are very small p_c-p_1 and $p_{c_1}-p_c$ will depend entirely on the action of the gas within the layer of thickness s immediately adjacent to the surface. In these cases, however, *when* $\frac{c_1}{s}$ *or* $\frac{c_1-c_2}{s}$ *are small,* the action within this layer may be easily expressed.

114. Let the temperature of the internal surface (sphere or cylinder) be such that the mean value of α for the molecules which rebound from this surface (considered as a group in a uniform gas) is α_{c_1}; while the temperature of the external surface is such that the mean value of α for the molecules which rebound is α'.

The condition that $\frac{c_1}{s}$ or $\frac{c_1-c_2}{s}$ are small necessitates that the molecules which come up to the inner surface arrive as from a uniform gas such that $\alpha=\alpha'$. That is to say, none of the molecules which rebound from the inner surface can return until their characteristics have been completely modified by the external surface. For if $\frac{c_1-c_2}{s}$ is small the molecules will cross the interval between the surfaces without encounter, while if $\frac{c_1}{s}$ is small, although $\frac{c_1-c_2}{s}$ may be large, the characteristics of the gas will be but slightly affected by the internal layer at a distance s from that surface, and, by theorem II., the approaching molecules will arrive as from a uniform gas in the mean condition of the gas at a distance s.

I shall first consider the case in which $\frac{c_1-c_2}{s}$ is small.

The number of molecules which arrive at the inner surface is proportional to $\rho'\alpha'$, and the number which rebound is proportional to $\rho_c\alpha_c$, and since the numbers must be the same we have

$$\rho'\alpha'=\rho_c\alpha_c$$

The momentum imparted to the surface by the incident molecules is $\frac{1}{2}\frac{\rho'\alpha'^2}{2}$, and that imparted by the rebounding molecules is $\frac{1}{2}\frac{\rho_c\alpha_c^2}{2}$, therefore

$$p_{c_1}=\tfrac{1}{4}\rho'\alpha'(\alpha_{c_1}+\alpha') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

Since the molecules which rebound from the internal surface all proceed to the external surface, and the surfaces are concentric, we have

$$p'=\frac{\rho'\alpha'^2}{2}+\frac{c_1^2}{c_2^2}\frac{\rho'\alpha'}{4}(\alpha_{c_1}-\alpha') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (137)$$

Therefore

$$p_c-p'=\frac{\rho'\alpha'}{4}\frac{c_2^2-c_1^2}{c_2^2}(\alpha_c-\alpha')$$

or

$$\frac{p_c-p'}{p}=\frac{1}{2}\frac{c_2^2-c_1^2}{c_2^2}\frac{\alpha_c-\alpha'}{\alpha'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

Equation (138) holds whatever may be the value of $\frac{c_1}{s}$ provided $\frac{c_2-c_1}{s}$ is small, and it also holds when $\frac{c_2-c_1}{s}$ is large, provided $\frac{c_1}{s}$ is small. When $\frac{c_1}{s}$ is small and $\frac{c_2-c_1}{s}$ is large c_1 may be neglected in comparison with c_2, and we have

$$\frac{p_c-p'}{p}=\frac{1}{2}\frac{\alpha_c-\alpha'}{\alpha'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

Equation (139) is almost identical with what equation (132) becomes as s approaches in value to r. If $s=r$, then the only difference in those two equations is in the coefficient. In comparing these equations, however, it must be noticed that in (132) α is not the same as α_c, for α_{c_1} only refers to the one set of molecules—those which are receding from the surface, whereas α refers to both sets.

At the surface when either $\frac{c_1}{s}$ or $\frac{c_2-c_1}{s}$ are small

$$\alpha=\frac{\alpha_{c_1}+\alpha'}{2}.$$

Whence making this substitution in equation (132), and putting $s=r$ the coefficient differs from that in equation (139) by $\frac{8}{\pi}$, which shows the extent to which discontinuity at the surface affects the result.

General equation of impulsion.

115. From equations (132) and (139) we may form an equation which will hold for all values of $\frac{c}{s}$.

For if the surfaces are spherical

$$\frac{p_c-p'}{p'}=\left\{\frac{1}{2}\frac{c_2^2-c_1^2}{c_2^2}f_5\left(\frac{c_1}{s},\frac{c_2}{s}\right)+\frac{8}{\pi}\frac{s^2}{c_1^2}f_6\left(\frac{c_1}{s},\frac{c_2}{s}\right)\right\}\frac{\sqrt{\frac{\tau_c}{M}}-\sqrt{\frac{\tau'}{M}}}{\sqrt{\frac{\tau'}{M}}} \quad . \quad . \quad . \quad (140)$$

And for cylindrical surfaces

$$\frac{p_c-p'}{p^1}=\left\{\frac{1}{2}\frac{c_2-c_1}{c_2}f_5\left(\frac{c_1}{s},\frac{c_2}{s}\right)+\frac{4}{\pi}\frac{s^2}{c_1{}^2}\frac{1}{\log_e c_1}f_6\left(\frac{c_1}{s},\frac{c_2}{s}\right)\right\}\frac{\sqrt{\frac{\tau_c}{\mathrm{M}}}-\sqrt{\frac{\tau'}{\mathrm{M}}}}{\sqrt{\frac{\tau'}{\mathrm{M}}}} \quad . \; . \quad (141)$$

where $f_5\left(\frac{c_1}{s},\frac{c_2}{s}\right)$ and $f_6\left(\frac{c_1}{s},\frac{c_2}{s}\right)$ are respectively unity and zero when either $\frac{c_1}{s}$ or $\frac{c_1-c_2}{s}$ are zero, and respectively zero and unity when both $\frac{c_1}{s}$ and $\frac{c_2-c_1}{s}$ are infinite.

Equations (140) and (141) have been obtained on the assumption that the solid surfaces are either concentric spheres or concentric cylinders. But these equations indicate what would be the difference of pressure consequent on a difference of temperature whatever may be the shape of the surfaces, and particularly so when $\frac{c_1}{s}$ and $\frac{c_2-c_1}{s}$ are finite, which are the most important cases.

SECTION XII.—APPLICATION TO THE EXPERIMENTS WITH THE FIBRE OF SILK AND THE RADIOMETER.

116. Comparing the equations (140) and (141) with the equation of transpiration (101), it appears at once that when Ω is zero these equations are identical in form. Hence the curves expressing the relation between the impulsive forces and the density of the gas under any given conditions would be of the same character as those expressing the relation between the inequalities of pressure and density in the case of thermal transpiration through a particular porous plate, and it is not necessary for me again to examine this relation.

Besides which, the experiments on impulsion, elaborate as they have been, furnish nothing like the definite results which I have obtained in the experiments on thermal transpiration.

117. The principal results to be deduced from experiments other than those which are contained in this paper, are :—

(1.) That the force and motion are proportional to the difference of temperature, which results are seen to follow directly from equations (124) and (140).

(2.) That with a particular instrument the forces increase with the rarefaction up to a certain point, after which they fall off; this result also follows directly from the equation (140).

118. Equations (124) and (140) first revealed to me the fact that the pressure of gas at which the force would become appreciable must vary inversely as the size of the surface.

From equation (140) it appears that up to a certain point

$$p_c-p' \propto \frac{ps^2}{c_1{}^2}$$

and since $s \propto \frac{1}{\rho}$ and $p \propto \rho$ it appears that

$$p_c - p' \propto \frac{1}{c_1^2 \rho}.$$

So that with gas at a given density the smaller the surface the greater would be the intensity of the impulsive force; and hence I was led to try the fibre of silk, with which I obtained evidence of the force at densities of half an atmosphere; whereas in the radiometer, with vanes something like 500 times as broad as the fibre of silk, the force does not manifest itself until the density is very small indeed.

Earlier conclusions.

119. The equations (124) and (140) show that both the forces and the consequent motion are, *cæteris paribus*, proportional to the heat communicated from the surface to the gas; for by equation (128) $\alpha_c - \alpha' \propto \mathrm{H}$ where H is proportional to the heat communicated from the surface to the gas.

The necessity of such a relation was the subject of my former paper.* I then obtained the formula

$$f = \epsilon \sqrt{\frac{d}{3gp}}.$$

To translate this into the symbols of the present paper

$$f = p_c - p',$$

$$d = g\rho$$

and

$$\epsilon = \sqrt{\frac{3}{2}} \frac{\sqrt{\pi}}{18} \frac{\mathrm{H}}{c^2}.$$

According to my intention ϵ should have been equal $\frac{\mathrm{H}}{c^2}$, but from the manner in which it was obtained it has the value given above (Appendix, note 5 (b)). Hence we have

$$p_c - p' = \frac{\sqrt{\pi}}{18} \frac{\mathrm{H}}{c^2 \alpha}.$$

The corresponding equation (Appendix, note 5 (a)) derived from equation (140) is

$$p_c - p' = \left\{ \frac{\sqrt{\pi}}{18} f_5\left(\frac{c_1}{s}, \frac{c_2}{s}\right) + \frac{8}{11} \frac{s}{\sqrt{\pi}} \frac{1}{c_1} f_6\left(\frac{c_1}{s}, \frac{c_2}{s}\right) \right\} \times \frac{\mathrm{H}}{c^2 \alpha},$$

or when $\frac{c}{s}$ is small

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* Proc. Roy. Soc., 1874, p. 407.

$$p_c - p' = \frac{\sqrt{\pi}}{18} \frac{\mathrm{H}}{c^2 \alpha},$$

and when $\frac{c}{s}$ is large

$$p_c - p' = \frac{8}{11} \frac{1}{\sqrt{\pi}} \frac{s}{c} \frac{\mathrm{H}}{c^2 \alpha}.$$

It thus appears that the present results entirely confirm the previous results so far as they went; and the present investigation is a completion, not a correction, of the former one.

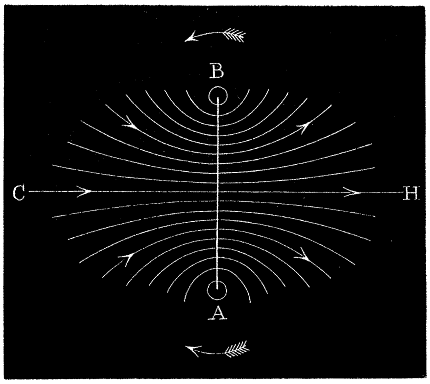
The present investigation shows that, besides being proportional to the quantity of heat, the force is proportional to the linear divergence of the lines along which the heat flows; and hence, if these lines are parallel, no matter how great may be the difference of temperature, the gas can exert no pressure above the normal pressure which it will exert on all surfaces alike. This is the case, whether the heat is communicated to gas or is spent in causing evaporation from the surface.

The relation between the difference of pressure and the divergence of the lines of flow affords a clear explanation of the complex phenomena of the radiometer; and as these phenomena have attracted a great deal of interest, I feel that an explanation of them will not be out of place.

Divergence of the lines of flow and the radiometer.

120. We may readily obtain a graphic representation of the results expressed by equations (124) and (140).

Fig. 12.



Let A B, fig. 12, be a plate from which heat is being communicated to the surrounding gas. Then the lines representing the flow of heat, drawn according to the law of conduction, are shown in the figure.

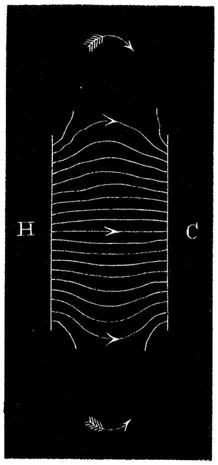
(1.) The shape of these lines depends on the distribution of temperature over A B.

Fig. 12 shows what the lines would be if A B were hot on one side and cold on the other, the gas being at the mean temperature and of unlimited extent.

(2.) The distribution of temperature on an opposite surface, or containing vessel, will also affect the shape of the lines of flow.

Fig. 13 shows the lines between two parallel plates opposite one another, the inside face, H, being hotter than the opposite face, C, while the gas and the outside faces of the plates are at the mean temperature of C and H.

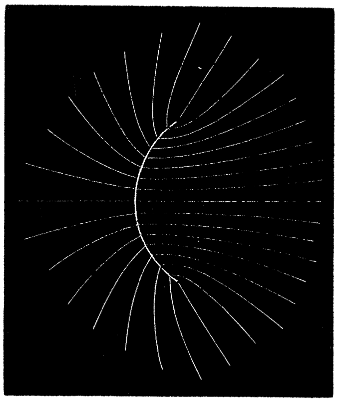
Fig. 13.



(3.) The shape of the lines will also depend on the shape of the hot surface, and the nature of the surface as affecting the rate at which it communicates heat to the gas.

Fig. 14 shows the direction of the lines for a cup-shaped surface, supposed to be uniformly at a higher temperature than the gas.

Fig. 14.



In all these figures the lines are supposed to be drawn so that the distance between any two lines is somewhere between s and $2s$, so that the excess of pressure along the lines of flow depends, *cæteris paribus*, on the angle between two consecutive lines. Thus the divergence of the lines indicates the excess of pressure, the excess being, *cæteris paribus*, proportional to the square of the angle of divergence.

The shapes of the curves of flow are independent of the density of the gas, but the distance between these lines varies inversely as the density; and since the angle between the lines at distance s increases with s, we see that the excess of pressure along the lines of flow increases as the density diminishes, as long as the mean range of the molecules is not limited by the size of the containing vessel. When this point is reached, there can be no further increase in the mean range, and the excess of pressure will pass through a maximum value, and then fall with the density, until the ratio of the excess of pressure to the mean pressure becomes constant, which it will be in the limit.

The distribution of the force of impulsion as indicated by the figures.

121. In fig. 12 the divergence of the lines of flow is much greater towards the edges of the plates than in the centre; hence the excess of pressure will be greater towards the edges. In the same way, on the cold side of the plate, the convergence of the lines of flow is greatest towards the edges, and here the pressure will be least.

When the density of the gas is such that the width of the plate is large compared with s, the divergence of the consecutive heat-lines at the middle of the plate is small, which shows that there would be but little action on this part of the plate. At the edges, however, the divergence is greater, and there must always be action at the edges; and the smaller the density of the gas, or the narrower the plate, the more nearly to the middle of the plate will the inequality of pressure extend. Thus with a very narrow plate, such as a spider-line, we may have the inequality of pressure all over the plate, although in the same gas, with a broad plate, the action might only extend to a distance from the edge equal to the thickness of the spider-line.

Fig. 13 illustrates the paradox which furnished the clue to this theory. Towards the middle of the plate the heat-lines are parallel, and consequently the pressure would be equal and opposite on both plates, being the mean pressure of the gas; so that, if the plates were of unlimited extent, there would be no change in the pressure on either plate due to the one being hot and the other cold.

At the edges, however, the heat-lines diverge from the hot plate; hence at this point this plate would be subject to an excess of pressure, which would tend to force the plate back against the mean pressure of the gas on the outside. At the edges of the cold plate the heat-lines converge on to the plate; hence there will be a deficiency of pressure, and the tendency will be for the pressure at the back to force the plate forward toward the hot plate. Thus the action is not to separate the plates, but to force them both to move in the direction of the hotter plate—to cause the hot plate to run away, and the cold plate to follow it.

Fig. 14 shows the inequality of pressure which may exist over a surface, itself at uniform temperature, but differing from the temperature of the gas.

On the convex side the lines diverge much more rapidly than on the concave side, and hence the inequality of pressure due to the communication of heat will be greater on the convex side.

Stability of the equilibrium.

122. The figures give the lines of flow on the supposition that the gas is at rest and the surfaces all rigidly fixed. The condition would then be one of equilibrium. But in order that such a condition might be maintained, it would be necessary that the condition should be one of stable equilibrium. This is a point on which the foregoing reasoning furnishes us with no information.

It is satisfactory, therefore, to be able to see what must happen if the equilibrium is unstable. This is shown by equation (124), which gives the motion of the gas, so that there may be no forces.

If either the surface A B, or the containing vessel, be perfectly free to move, then no inequality of pressure will be possible, but motion must ensue. Equation (124) shows the law of such motion.

The motion.

123. The motion is given by

$$\bar{u}=\frac{s}{\sqrt{\pi}}\frac{d\alpha}{dx}.$$

If we suppose the containing vessel to be fixed, then, to allow of the motion of the gas, the plate must move with the gas. On the other hand, if the plate be held, the vessel will be carried by the gas in the opposite direction. Such is the phenomena of the radiometer. The vanes are as nearly as possible free to move in the vessel, so that their motion merely indicates the motion of the gas caused by the inequality of temperature in the gas, which inequality is maintained by the unequal temperature of the two sides of the vanes arising from their different power of absorbing light, or, in the case of curved vanes, by the greater temperature of the vanes as compared with the vessel.

The constraint which is put upon the vanes in a rotatory manner necessarily disturbs somewhat the free motion of the gas, as must also the friction of the pivot and other resistances. Therefore the condition of the gas within the vessel cannot be one of absolutely equal pressure; and when either the size of the vanes or the density of the gas are too great, the utmost inequality of pressure is insufficient to overcome these resistances, and there is no motion. If, then, exhaustion proceeds, the inequality of pressure increases, and motion ensues—the rate of which, if the vanes were absolutely free, would increase as the density diminished, until the mean range was limited by the size of the envelope, so that the larger the envelope the greater the possible rate of motion. When the paths of the molecules are limited by the size of the vessel, the motion would, if the vanes were perfectly free to move, remain constant for all further exhaustion ; but the inequalities of pressure which the gas is capable of exerting diminish with the further rarefaction, and hence, in time, must cease to be sufficient to overcome the resistances to which the motion of the vanes is subject, and then the motion ceases.

124. There are many other points about the phenomena of the radiometer, but with most of these I have already dealt in my former papers, the reasoning of which, so far as it goes, appears to me to be perfectly consistent with the more complete view of the action to which I have now attained.

My chief object in introducing the phenomena of the radiometer in this paper has been to bring out how completely impulsion belongs to the same class of actions as thermal transpiration, and the other phenomena depending on the relation which the size of the external objects bears to the mean range within the gas.

The action does not depend on the distance between the hot and cold plates.

It has been supposed by some writers on the radiometer, that the action depends essentially on the distance between the vanes and the sides of the vessel. This distance, however, is now seen not to be of primary consequence, as no action will result, however close the plates may be, unless they are of limited extent—of sizes comparable with the mean ranges.

SECTION XIII.—SUMMARY AND CONCLUSION.

125. The several steps in this investigation have now been described in detail. They may be summarized as follows :—

(1.) The primary step from which all the rest may be said to follow is the method of obtaining the equations of motion, so as to take into account not only the normal stresses which result from the mean motion of the molecules at a point, but also the normal and tangential stresses which result from a variation in the condition of the gas (assumed to be molecular). This method is given in Sections VI., VII., and VIII.

(2.) The method of adapting these equations to the case of transpiration through tubes or porous plates is given in Section IX. The equations of steady motion being reduced to a general equation, expressing the relation between the rate of transpiration, the variation of pressure, the variation of temperature, the condition of the gas, and the dimensions of the tube.

In Section X. is shown the manner in which were revealed the probable existence (1) of the phenomena of *thermal transpiration*, and (2) the law of correspondence between all the results of transpiration with different plates, so long as the density of the gas is inversely proportional to the lateral linear dimensions of the passage through the plate; from which revelations originated the idea of making experiments on thermal transpiration and transpiration under pressure.

(3.) The method of adapting the equations of steady motion to the case of impulsion is given in Section XI.

In Section XII. is shown how it first became apparent that the extremely low pressures at which alone the phenomena of the radiometer had been obtained were consequent on the comparatively large size of the vanes, and that by diminishing the size of the vanes similar results might be obtained at higher pressures; whence followed the idea of using the fibre of silk and the spider-line in place of the plate-vanes.

(4.) In Section XII. it is also shown that while the phenomena of the radiometer result from the communication of heat from a surface to a gas, as explained in my former paper, these phenomena also depend on the divergence of the lines of flow; whence it is shown that all the peculiar facts that have been observed may be explained.

(5.) In Section X. it is also shown that the phenomena of transpiration, resulting from a variation in the molecular constitution of the gas (investigated by GRAHAM), are also to be explained by the equation of transpiration.

(6.) Section II. (Part I.) contains a description of the experiments undertaken to verify the revelations of Section X. respecting *thermal transpiration;* which experiments establish not only the existence of the phenomena, but also an exact correspondence between the results for different plates at corresponding densities of the gas.

(7.) Section III. contains a description of the experiments on *transpiration under pressure,* undertaken to verify the revelations of Section X. with respect to the correspondence between the results to be obtained with plates of different coarseness at certain corresponding densities of the gas; which experiments proved, not only the existence of this correspondence, but also that the ratio of the corresponding densities in these experiments are the same as the ratio of the corresponding densities with the same plates for thermal transpiration—a fact which proves that the ratio depends on the relative coarseness of the plates.

(8.) Section IV. contains a description of the experiments with the fibre of silk and with the spider-line, undertaken to verify the revelations of Section XII.; from which experiments it appears that, with these small surfaces, phenomena of impulsion similar to those of the radiometer occur at pressures but little less than that of the atmosphere.

126. As regards transpiration and impulsion, the investigation appears to be complete. Most, if not all, the phenomena previously known have been shown to be such as must result from the tangential and normal stresses consequent on a varying condition of molecularly constituted gas; while the previously unsuspected phenomena to which it was found that a variation in the condition of a molecular gas must give rise, have, on trial, been found to exist.

The results of the investigation lead to certain general conclusions which lie outside the immediate object for which it was undertaken. The most important of these viz.: that gas is not a continuous plenum, has already been noticed in Art. 5, Part I.

The dimensional properties of gas.

127. The experimental results, considered by themselves, bring to light the dependence of a class of phenomena on the relation between the density of the gas and the dimensions of objects, owing to the presence of which the phenomena occur. As long as the density of the gas is inversely proportional to the coarseness of the plate, the transpiration results correspond; and in the same way, although not so fully investigated, corresponding phenomena of impulsion are obtained as long as the density of the gas is inversely proportional to the linear size of the objects exposed to its action. In fact, the same correspondence appears with all the phenomena investigated.

We may examine this result in various ways, but, in whichever way we look at it, it can have but one meaning. If in a gas we had to do with a continuous plenum such that any portion must possess the same properties, we should only find the same properties, however small might be the quantity of gas operated upon. Hence, in the fact that we find properties of a gas depending on the size of the space in which it is enclosed, and of the quantity of the gas enclosed in this space, we have proof that gas is not continuous—or, in other words, that gas possesses a dimensional structure.

In virtue of their depending on this dimensional structure, and having afforded us proof thereof, I propose to call the general properties of gas on which the phenomena of transpiration and impulsion depend, the *Dimensional Properties of Gases.*

This name is also indicative of the nature of these properties as deduced from the molecular theory; for by this it appears that these properties depend on the mean range—a linear quantity which, *cœteris paribus,* depends on the distance between the molecules.

In forming a conception of a molecular constitution of gas, there is no difficulty in realizing that such dimensional properties exist; there is, perhaps, greater difficulty in conceiving molecules so minute and so numerous that, in the resulting phenomena, all evidence of the individual action is lost. But the real difficulty is to conceive such a range of observational power as shall embrace, on the one hand, a sufficient number of molecules for their individualities to be entirely lost, while, on the other hand, it can be so far localized as regards time and space that, if not the action of individuals, the actions of certain groups or classes of individuals becomes distinguishable from the action of the entire mass. Yet this is what we have in the phenomena of transpiration and impulsion.

Although the results of the dimensional properties of gases are so minute that it has required our utmost powers to detect them, it does not follow that the actions which they reveal are of philosophical importance only. The actions only become considerable within extremely small spaces, but then the work of construction in the animal and vegetable world, and the work of destruction in the mineral world, are carried on within such spaces. The varying action of the sun must be to cause alternate inspiration and expiration of air, promoting continual change of air within the interstices of the soil as well as within the tissue of plants. What may be the effects of such changes we do not know, but the changes go on; and we may fairly assume that in the processes of nature the dimensional properties of gas play no unimportant part.

Nor is this all. It is by aid of the analogy which gas affords us that we must look forward to solve the mystery of the luminiferous ether. And although all attempts to frame a satisfactory hypothesis as to the molecular constitution of ether have hitherto failed, in none of these hypotheses have the tangential and normal stresses arising from a varying condition been taken into account; whereas the recognition of the part which these stresses play in the properties of gases shows, or at least suggests, the possibility that the phenomena of ether which we observe may depend largely upon analogous stresses.

APPENDIX.

(Added December, 1879).

NOTE 1.

Since the reading of this paper I have had my attention called to a paper by W. FEDDERSEN ("Über Thermodiffusion von Gasen," POGG. 'Ann.,' 1873). FEDDERSEN made some experiments, and seems to have thought that he had discovered some such phenomenon. But the results he obtained were attributed by M. J. VIOLLE to the presence of the vapour of water, against which no precautions appear to have been taken ('Journal de Physique,' 1875, p. 90). That M. J. VIOLLE was right there can be do doubt, for the results obtained are now seen to be much too large for the true results, and are similar to those which I obtained before I had succeeded in sufficiently drying the air.

NOTE 2.

GRAHAM applied the term "transpiration" to the passage of gases through capillary tubes as distinguished from the passage of gases through larger tubes and through apertures in thin plates, and applied the term "effusion" to the passage of gases through minute apertures in thin plates.

He did not apply either of these terms to the passage of gases through porous plates, because his experiments led him to conclude that the phenomena attending such passage were not the same as the phenomena attending either of the former, but were somewhere between the two.

By the fuller light thrown on to the subject by this investigation it appears that in the limit, when the tubes and holes are small enough according to the condition of the gas, the laws of transpiration are strictly the same as those of effusion, the theory of the phenomena being the same. Hence the continued use of two names appears to be unadvisable.

The term "transpiration" has been chosen in preference to "effusion," because it is found that as the passages become coarser, according to the condition of the gas, the law of the passage of gas through porous plates is still in strict accordance with the law of the passage through tubes, showing that the passages are of the nature of tubes rather than thin plates.

NOTE 3, ART. 7.

It will be observed that this dependence of the phenomena on a relation between the size of the surfaces and the mean path of a molecule is essentially different from what has been a common, but as is herein shown, erroneous supposition, that the phenomena essentially depend on distance separating the opposite surfaces. The one supposition makes the action of the radiometer depend on the size of the vanes, but leaves it independent of the size of the envelope, while the other makes the action depend on the size of the envelope, but leaves it so far independent of the size of the vanes.

NOTE 4, ARTS. 41 AND 104.

The assumption that the coefficients A, m, λ_1, and λ_2, also m' and λ_3, equation (99), are the same for stucco as for meerschaum is equivalent to assuming that the only respect in which the interstices of these plates differ is that of coarseness. There is no *à priori* ground for making this assumption. The fact that the logarithmic homologues for stucco fit those for meerschaum through such a considerable range of densities proves the approximate truth of the assumption; but it is possible, since c_s and c_m are arbitrary dimensions, that the curves for transpiration under pressure depending on A, m, λ_1, and λ_2 may approximately fit for one value of $\frac{c_s}{c_m}$, and the curves for thermal transpiration depending on A_1, m, λ_1, λ_2, m', and λ_3 may approximately fit for another value of $\frac{c_s}{c_m}$. If this were so log. $\left(\frac{c_s}{c_m}\right)$; the shift necessary to bring the curves into coincidence would not be the same for transpiration under pressure as for thermal transpiration, and as has been pointed out (Art. 41), this is to a certain extent the case, this ratio having the values 6·5 and 5·6—a difference which was sufficiently decided to call for notice, but which is not so large but that, as pointed out (Art. 41), it may possibly be due to the plates being hot in the one case and cold in the other. In any case the smallness of the difference is an additional proof that the interstices do not greatly differ as passages in any respect except that of size.

NOTE 5, ART. 119.

(*a.*)

$$\frac{\mathrm{H}}{c_1^2}=\sigma\{\mathrm{M}(u^2+v^2+w^2)\}.$$

Whence at the surface when $\frac{c_1}{s}$ is sufficiently small we have by equation (18)

$$\frac{\mathrm{H}}{c_1^2}=\frac{3}{2}\frac{\rho}{\sqrt{\pi}}(\alpha_c^3-\alpha'^3),$$

or neglecting $\left(\frac{\alpha_c-\alpha'}{p'}\right)^2$

$$\frac{\mathrm{H}}{c_1^2}=\frac{9p'}{\sqrt{\pi}}(\alpha_c-\alpha').$$

And when $\frac{c_1}{s}$ is large, we have by equation (128)

$$\frac{\mathrm{H}}{c^2}=\frac{11p'}{\sqrt{\pi}}\frac{s}{c_1}(\alpha_c-\alpha').$$

Therefore substituting for $\frac{\alpha_c-\alpha'}{\alpha'}$ in equation (140)

$$p_c-p_1=\left\{\frac{\sqrt{\pi}}{18}f_5\left(\frac{c_1}{s},\frac{c_2}{s}\right)+\frac{8}{11\sqrt{\pi}}\frac{s}{c_1}f_6\left(\frac{c_1}{s},\frac{c_2}{s}\right)\right\}\frac{\mathrm{H}}{c_1^2\alpha}.$$

(*b*.) In my former paper (Proc. Roy. Soc. 1874, p. 407)

$$\epsilon=\frac{d}{g}\,\frac{v^2\partial v}{6},$$

$$p=\frac{1}{3}\,\frac{d}{g}v^2,\ \text{and}\ \frac{d}{g}=\rho.$$

Therefore, since

$$p=\rho\frac{\alpha^2}{2},\ v=\sqrt{\tfrac{3}{2}}\,\alpha,\ \text{and}\ \partial v=\sqrt{\tfrac{3}{2}}\,\partial\alpha,$$

$$\epsilon=\sqrt{\frac{3}{2}}\,\frac{p}{2}\partial\alpha,$$

and putting $\partial\alpha=\alpha_c-\alpha'$,

$$\epsilon=\sqrt{\frac{3}{2}}\,\frac{\sqrt{\pi}}{18}\,\frac{\mathrm{H}}{c^2}.$$

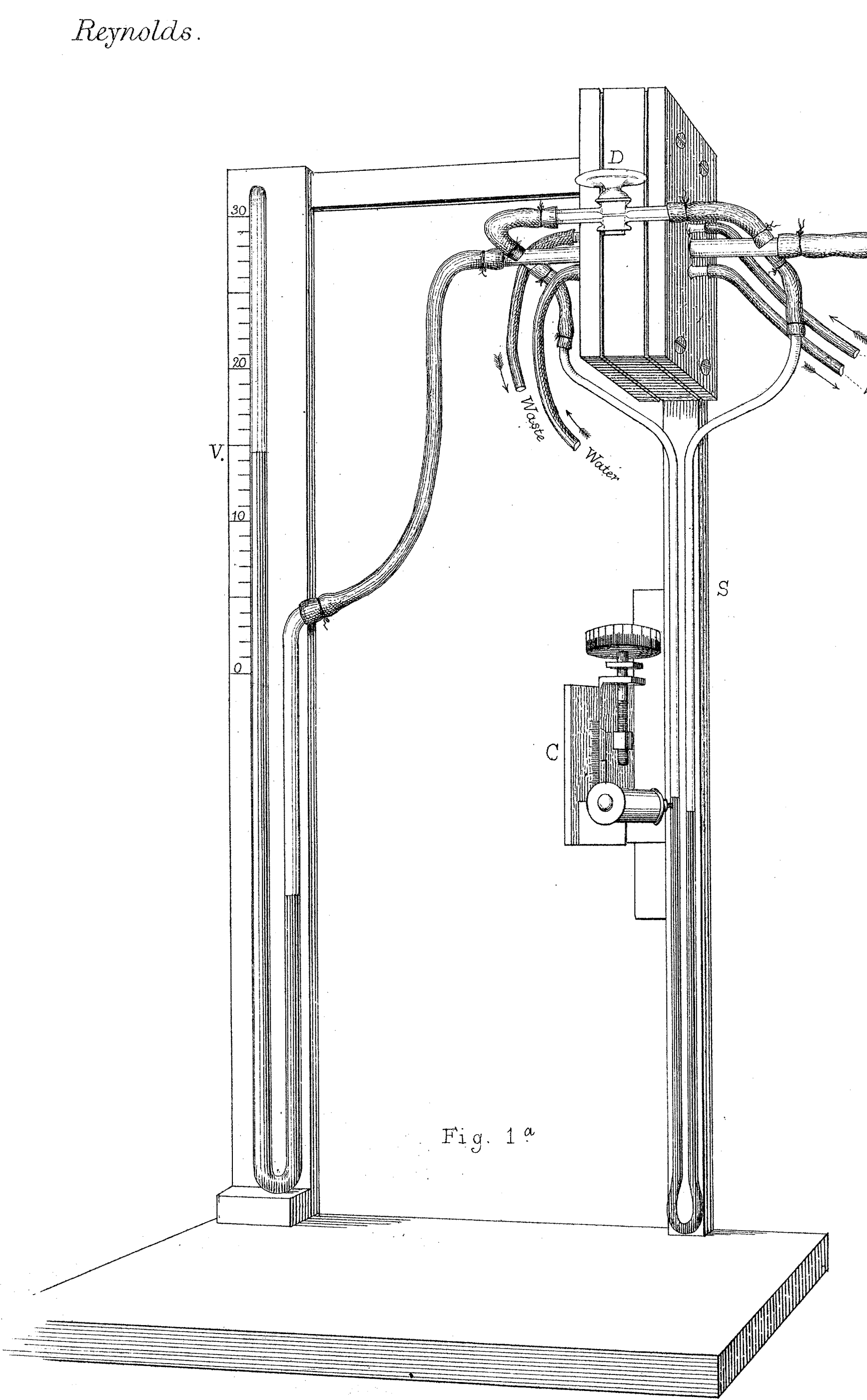
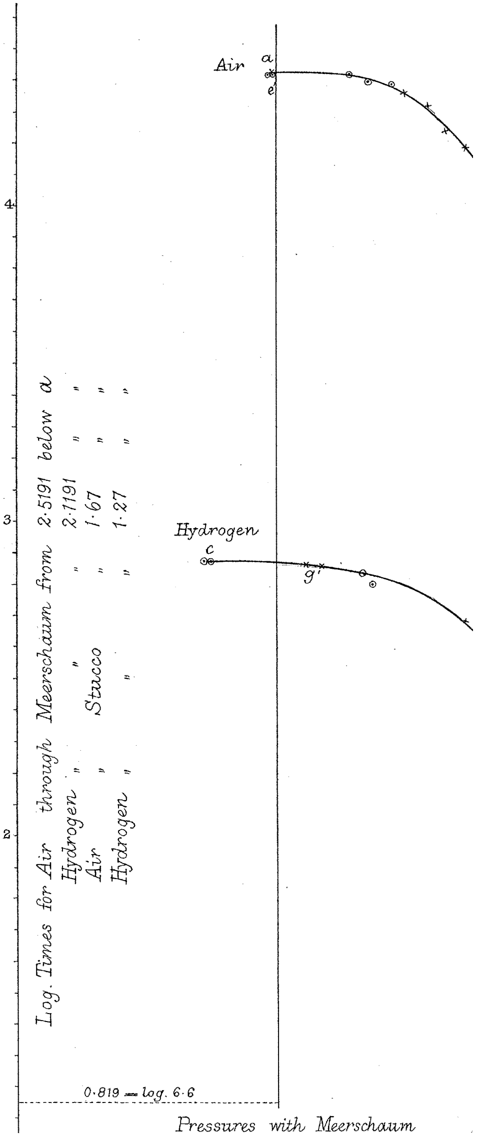


Fig. 1ª

Diagram I.



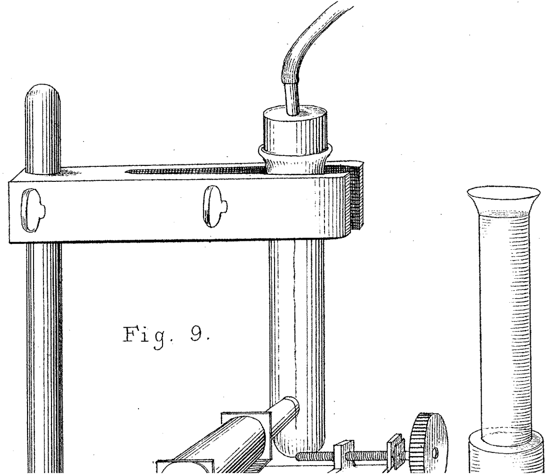
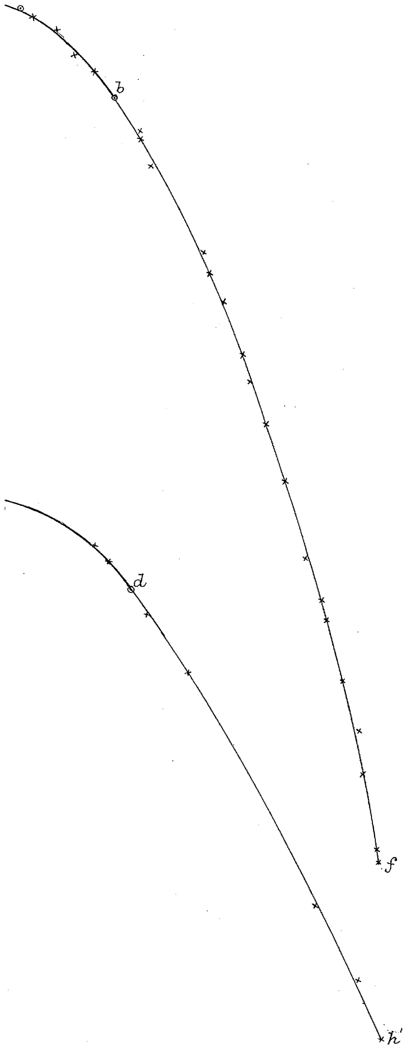
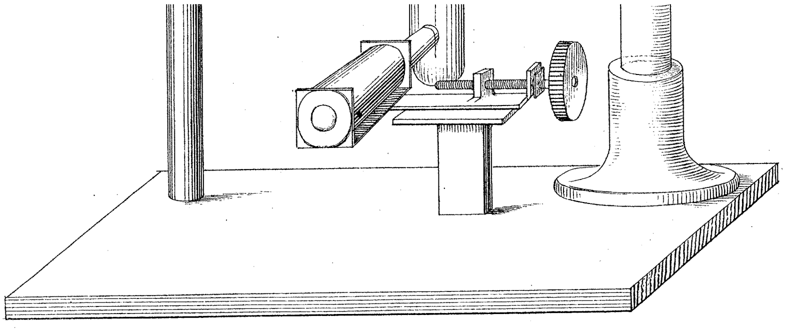


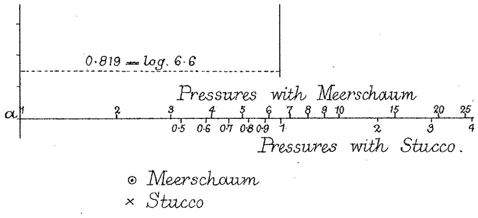
Fig. 9.

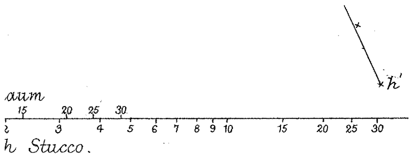
l. Trans: 1879. Plate 47.



aum

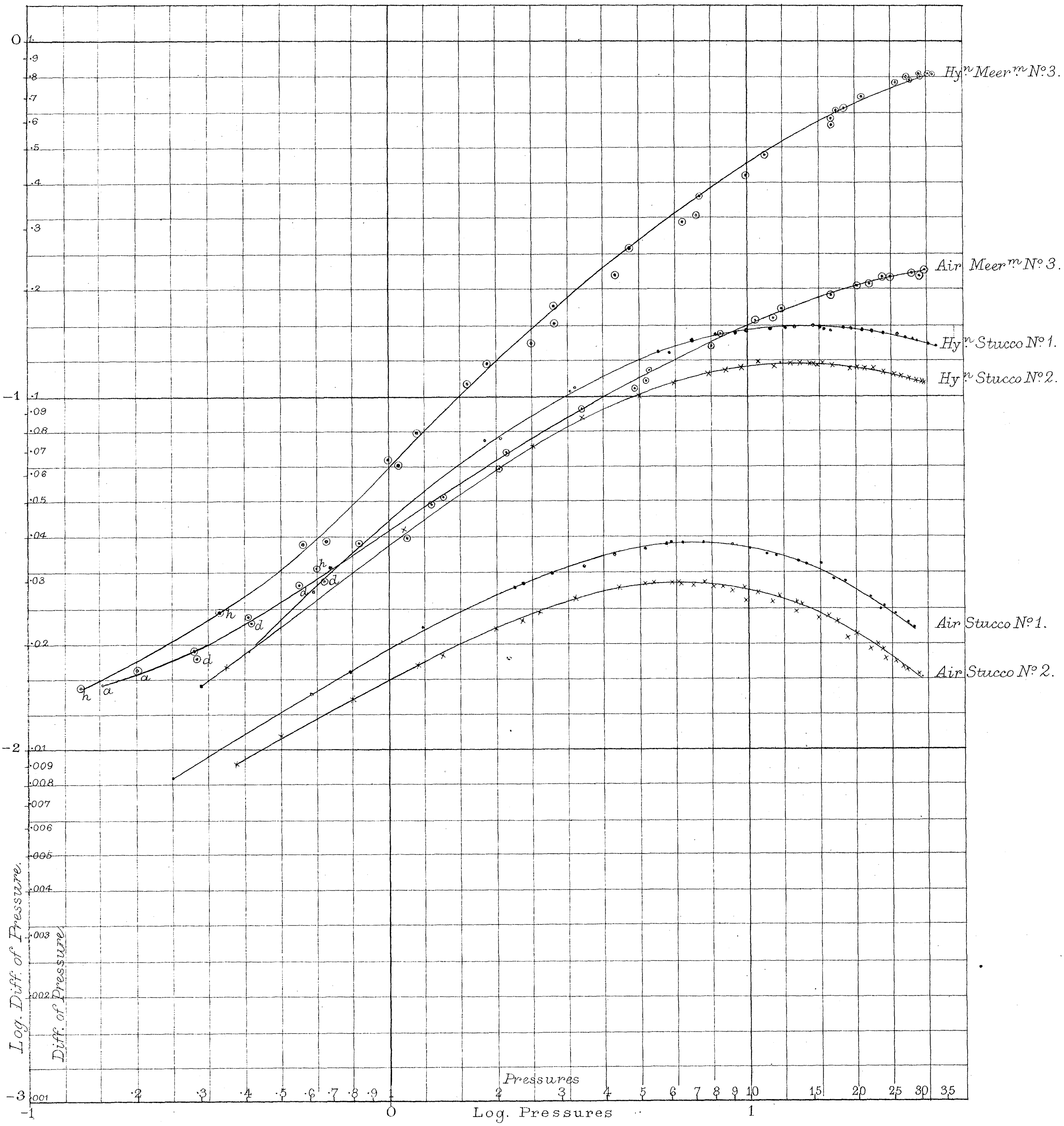



0·819 = log. 6·6
Pressures with Meerschaum
α
1
2
3
4
5
6
7
8
9
10
15
20
25
0·5
0·6
0·7
0·8
0·9
1
2
3
4
Pressures with Stucco.
⊙ Meerschaum
× Stucco



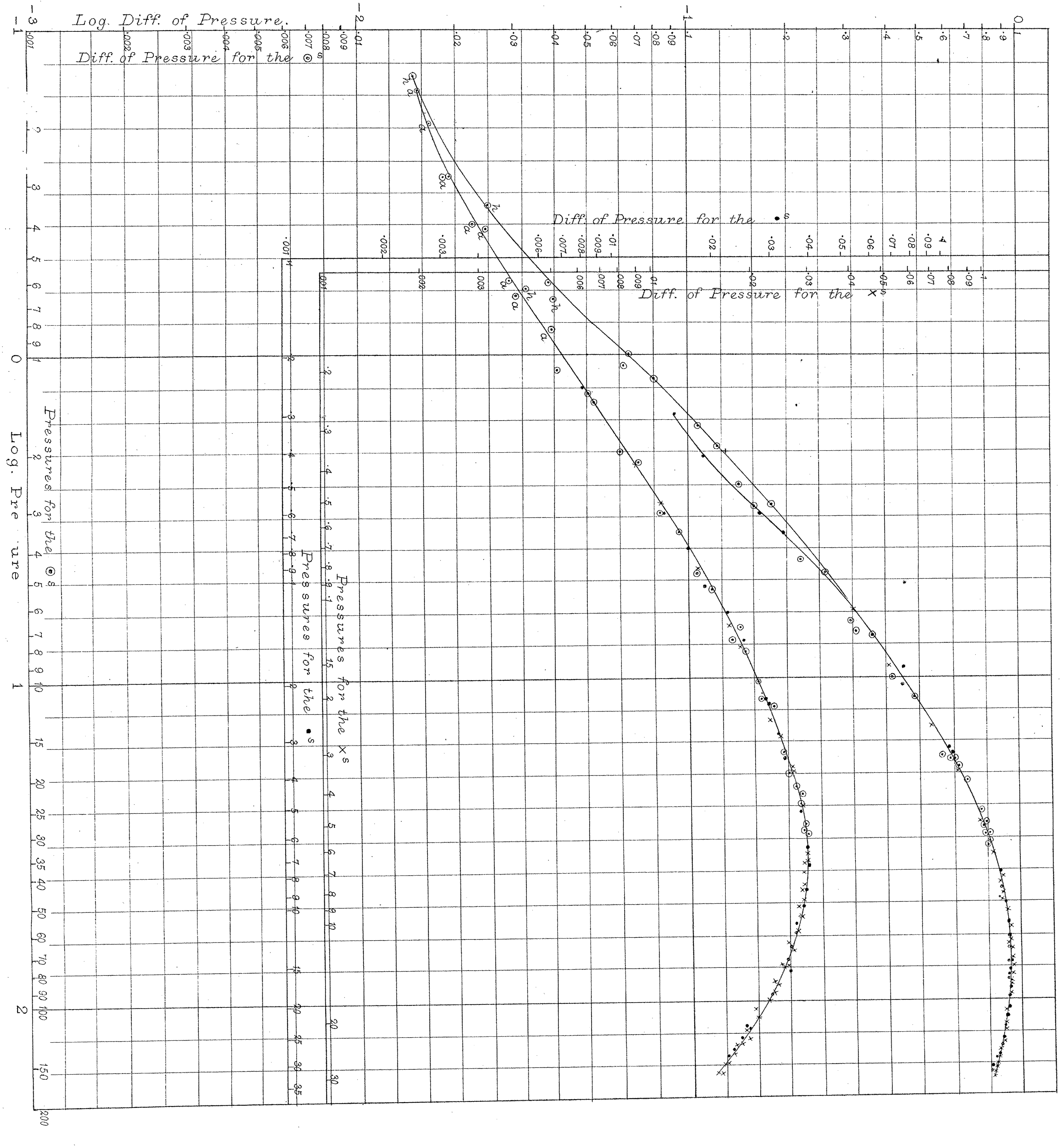
West Newman & C° lith.

Diagram 2.



West Newman & C^o lith.

Diagram 3.



West Newman & C^o. lith.

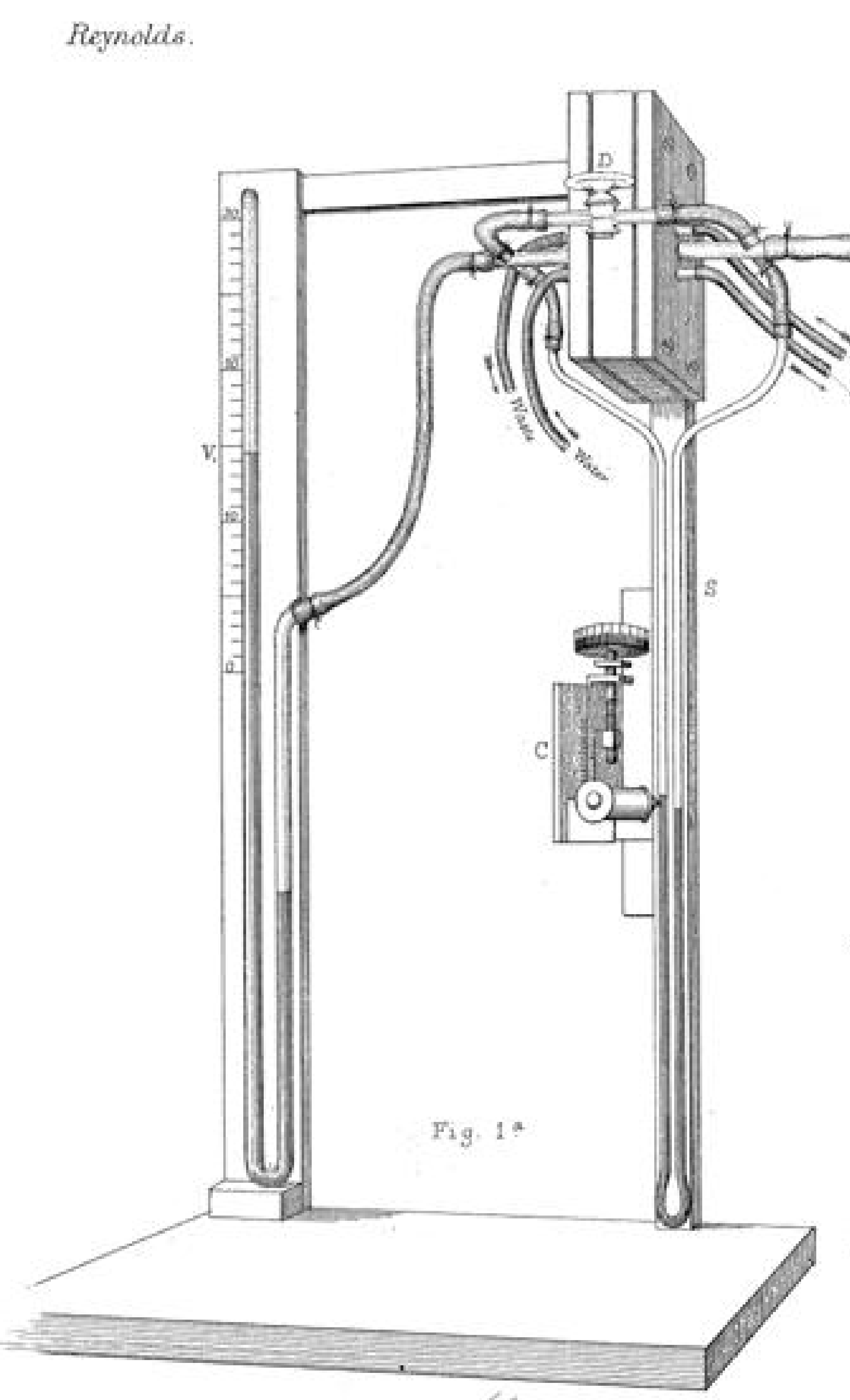


Fig. 1ª

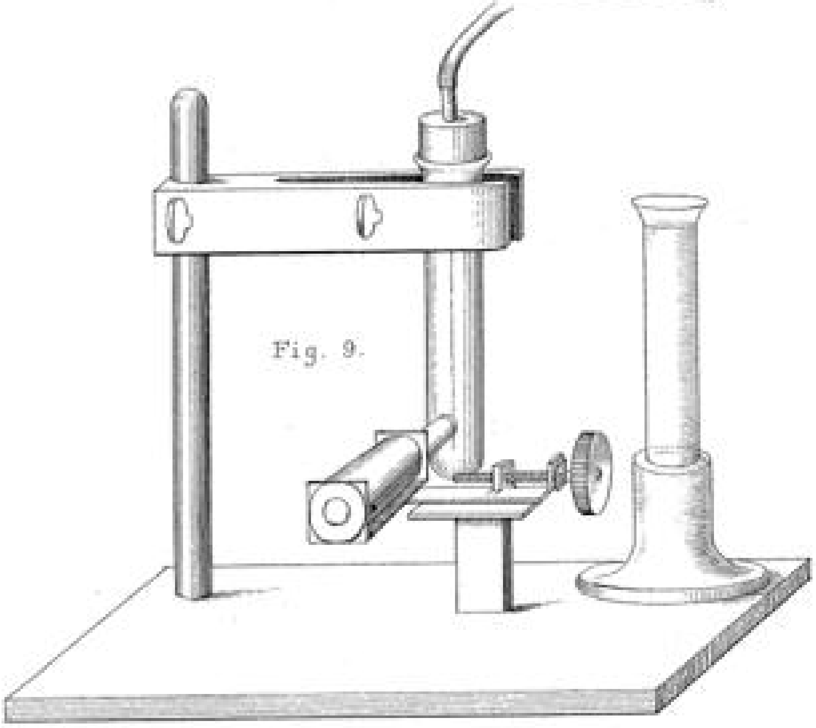
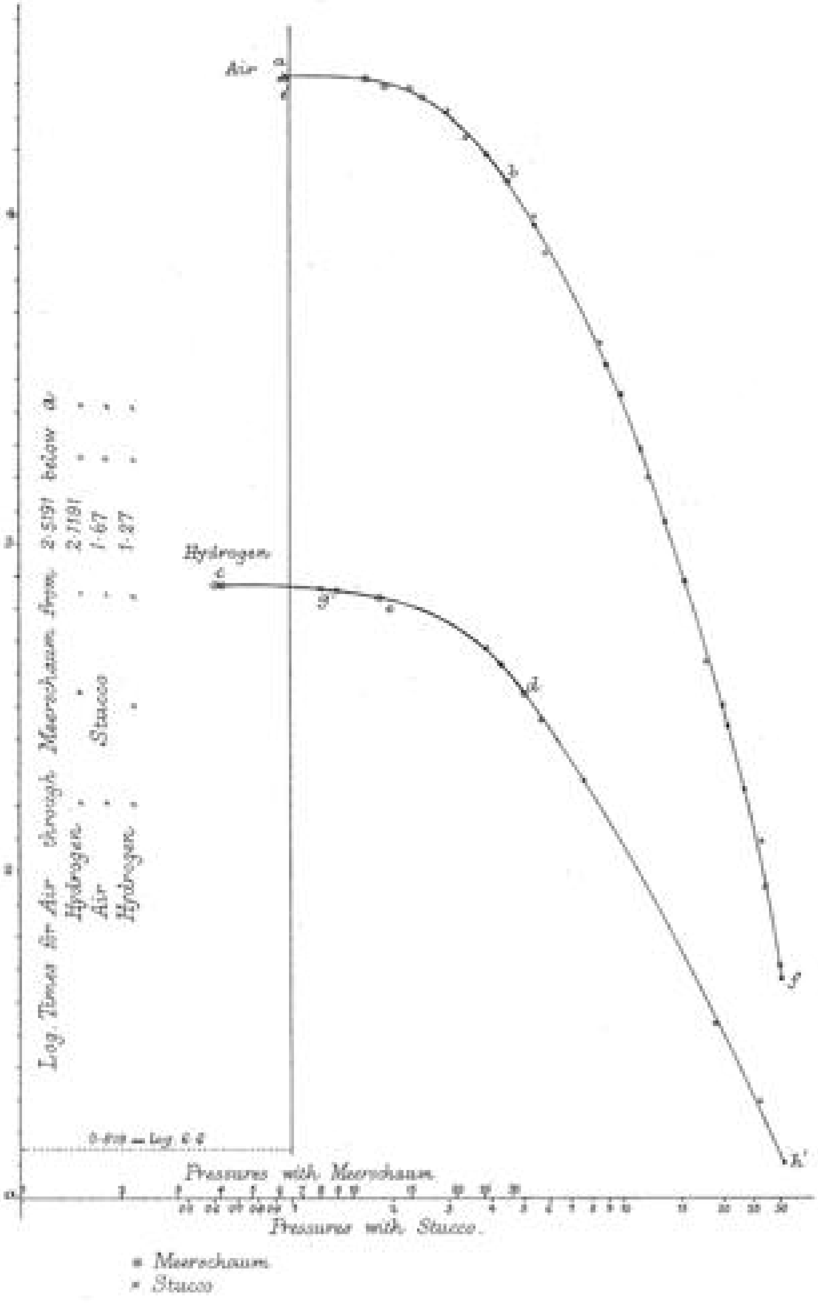


Fig. 9.

Diagram I.



West Newman & Cº lith.